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# From Grassmann, Riemann to Husserl: a brief history of concept of Manifold

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**Abstract**

Edmund Husserl's theory of manifold (*Mannigfaltigkeitslehre*) was formalized for the first time in his *Philosophie der Arithmetik*; in his *Logische Untersuchungen*, §§69–70; also discussed in *Ideen* I, §§72; in *Formale und Trascendentale Logik*, §§51–54; in *Logik und allgemeine Wissenschaftstheorie*, chapter two; and finally it appears in *Einleitung in die Logik und Erkenntnistheorie*, §§18–19. In each of these books, Husserl presents a concept of manifolds as an ontological form. Such form is necessarily axiomatic and appears as inspired by Bernhard Riemann's work. Indeed, Husserl, who studied and lectured extensively on Riemann's theories of space, presented his own conception of mathematics as a theory of manifolds as a generalization of Riemann's notion of manifold.

**Keywords**: Husserl, Riemann, Grassmann, Manifold, Phenomenology

## 1. Introduction

The goal of this paper is to take Husserl's first major work, *Philosophie der Arithmetik* (1891), as the starting point of our study on theory of manifold[1]. We articulate the claim that Husserl, using his classification of sciences, drove his mathematical work by a unitary philosophical program. Husserl's program includes several sections belonging to Ms. K I A, K I 15 y K I 4/9a-18a, most of them published in *Studien zur Arithmetik und Geometrie* (Hua XXI). In addition, we examine the ramifications of this concept in several areas of Phenomenology of Mathematics that have been the subject of recent commentaries and publications.[2] At the end, we conclude with a review of the second of two major themes in the aforementioned work of Husserl which have been the matter of a number of contemporary studies.

## 2. Husserlian manifolds or Riemannian manifolds?

The theoretical core of Husserl's theory of manifolds is dual. (*i*) It is a theory of theories, anchored in the German tradition of *Wissenschaftslehre*, guided by Fichte and Bolzano, and linked to the ancient tradition of *mathesis universalis*, explored by Descartes and Leibniz. (*ii*) The theory of manifolds is also a formal theory of everything (Milkov 2005). In fact, the theory of everything is intrinsically connected with the theory of theories.

Regarding this, Husserl refers to Riemann's conception of manifold and his generalization of geometrical theory, and takes his own notion of manifold as a kind of generalization of that of Riemann. But Riemann's influence on Husserl is not limited to the notion of mathematical manifold or to his views on pure mathematics. In fact, as attested by a posthumously published Husserl's book containing mostly material from the transition period of 1889–189 (edited as *Studien zur Arithmetik und Geometrie*, Hua XXI), Husserl not only extracts from Riemann his interest on the relationship between geometry and physical space, but also finds in Riemann the seed of his philosophy of mathematics as a whole (Rosado Haddock 2017).

It is indeed possible to show that Husserl's conception of mathematics as a theory of manifolds is a generalization and development of Riemann's views on mathematical knowledge and philosophy of mathematic. To clarify this statement, I will quote two passages from 1913 and 1901. The first one is from text No. 5 entitled "*Zwei Fragmente zum Entwurf einer Vorrede zur Zweiten Auflage der Logischen Untersuchungen*" (September 1913). In this "draft", Husserl presents a summary of his mathematical knowledge to the day and of his conversion from a philosophy of arithmetic (or philosophy of calculation), grounded on cardinal numbers to a *mathesis universalis*:

> Als ich aber daran ging, aufgrund der neuen Erkenntnis und unter Mithilfe Bolzanos meine logischen Vorlesungen völlig neu zu gestalten, erkannte ich das Unvollkommene des Bolzano'schen Entwurfs. Ihm fehlte die Idee einer rein formalen Mathematik bzw.

> "Mannigfaltigkeitlehre", die ich mir durch sachliche und historische Studien in einer Reinheit ausgebildet hatte, welche damals den Mathematikern noch keineswegs wie gegenwärtig vertraut war; und demgemäß (fehlte) auch jede Ahnung der inneren Einheit der formalen Logik mit der reinen Anzahlenlehre, der reinen Ordinalzahlenlehre, reinen Grössenlehre usw., schließlich der reinen Mannigfaltigkeits und Theorienlehre. (Hua. XX/1, 298)

A decade before, in September 7th 1901, Husserl sends a letter to Paul Natorp, in which confesses his acceptance of the Riemann's views on geometry and clearly states that accepts the existent of a diversity of geometric manifolds of *n*-dimensions, and, on the other hand, accepts that physical space, whose structure –curvature and dimensionality– should be empirical (Haddock 2017), i.e., the young Husserl not only did consider physical space as a particular case of the much more general concept of an *n*-fold extended magnitude, but also that the determination of the exact nature of physical space is not *a priori* available, but only empirically:

> In der Zeit von 1886/93 habe ich mich um die Theorie der Geometrie, der formalen Arithmetik u. Mannigfaltigkeits ehre sehr viel, periodenweise mit ausschließlicher Hingabe, bemüht. Davon giebt die Vorrede meiner Philosophie der Arithmetik 1891 entfernte Kunde (cf. den Hinweis auf Gauss' Anzeige zur 2. Abhandlung über biquadratische Reste, W.W. Bd. III), und zwar auch Kunde von manchen wichtigen Berührungen mit Ihren Ueberzeugungen. Auch ich fasste, beeinflußt durch Grassmann's Ausdehnungslehre und Gauss' Einführung der gemeinen complexen Zahlen (1. c.), die Ebene als eine gewisse stetige Doppelreihe, den Raum als eine gewisse stetige 3fache Reihe u.sw. In den gemeinen complexen Zahlen (bezw. auch in der Darstellung re') suchte ich die adäquaten arithmetischen Ausdrücke für die Ordnungsverhältnisse der Ebene nachzuweisen und ebenso in entsprechenden complexen Zahlen höherer Ordnung die arithmetischen Ausdrücke für die ebenen Mannigfaltigkeiten höherer Ordnung. (Hua Dok III/5, 80)

In light of the passages just cited, concepts such as "pure theory of magnitude" (*reinen Grössenlehre*) and "pure theory of manifold" (*reinen Mannigfaltigkeits*) become a clear theoretical background for Husserl in the period 1893-1900. Accordingly, there are two important facts that we will now be discussing in more detail: (1) The first fact that probes a linkage between Riemann's and Husserl's philosophical positions is that, by an appeal to mathematical practice, both theories gain a plausibility. This is important only because in their approaches to mathematical knowledge, the “space lived experience” is the starting point. So, neither Husserl nor Riemann began by asking about the relation between mathematical objects and space, but arrived at this question in an effort to provide an explanation for mathematical knowledge. (2) The second fact, is that Husserl states that the thesis about the Euclidean structure of physical space is an unfounded hypothesis made by natural scientists, which can only be founded empirically. Certainly, Riemann entitled his lecture *On the hypotheses which lie at the foundations of geometry* because he wants to specify that the properties that distinguish space from other conceivable 3-manifolds are only to be established *from* experience. The experience confirms that physical space is Euclidean, but these matters of fact are not necessary, but only of empirical certainty; they are hypotheses and not axioms (Ferreirós 2006, 75). So, Riemann and Husserl look upon the practice of mathematics not as the employment of fruitful techniques but as the collecting of *lived experiences* on space and the unity of sciences.

Accordingly, Husserl explained his views on logic and mathematics in the last chapter of the *Prolegomena zur reinen Logik* in terms that prove the objectivity of mathematics, i.e., of the development of his thought on our mathematical experience. Nevertheless, my reading of the young Husserl suggest that this account will be only understandable through references to Gauss[3] Bolzano and particularly Grassmann; mathematicians whose revolutionary discoveries changed the vision of Husserl in these early years.

## 3. Grassmann's *Ausdehnungslehre* and Husserl's *Philosophie der Arithmetik*

The purpose of this section is twofold. First, I will provide an explanation on Grassmann’s mathematical positions to evaluate Husserl’s original mathematical positions. Secondly, I will use this explanation as methodological clue to reconstruct the relationship between Husserl and Riemann in next section.

In *Über der Begriff der Zahlen* and in *Philosophie der Arithmetik*, emerges the idea that all philosophy of mathematics must start with the analysis of the concept of number based on the operations of collecting and counting; the "usual" definition: the "number is a multiplicity of unities" is formalized to a "formal reduction of calculate" (*rechnerischformelle Reduktion*).[4] In the second part of *Philosophie der Arithmetik*, chapters 10 and 11, after 200 pages of detailed *psychological* analysis Husserl realized that:

> Allzu voreilig liessen wir uns von der gemeiniiblichen und naiven Ansicht leiten, die den Unterschied zwischen symbolischen und eigentlichen Zahlvorstellungen nicht beachtet und der fundamentalen Tatsache nicht gerecht wird, dass alle Zahlvorstellungen, die wir über die wenigen ersten in der Zahlenreihe hinaus besitzen, symbolische sind und nur symbolische sein konnen; eine Tatsache, welche Charakter, Sinn und Zweck der Arithmetik ganz und gar bestimmt. (Hua XII, 190)

Indeed, if we allow ourselves to be guided by the common sense, which does not take into account the distinction between *inauthentic* and *authentic* representations of numbers, then we do not make justice to the fundamental fact that all number representations that we possess, beyond the firsts natural numbers series, are *symbolic* and *can* only be symbolic. This is what Husserl found particularly troublesome in the psychological analysis on representations of number. The psychological methodology did not allow to build bases prior to checking the ground of the bigger numbers; of course, the problem was that only small numbers and very easy arithmetical calculations are directly given to us, and thus are analyzable in terms of the first part of the *Philosophie der Arithmetik*, i.e., psychologically. His next approach is based on the distinction between authentic and symbolic concepts:

> Eine symbolische oder uneigentliche Vorstellung ist, wie schon der Name besagt, eine Vorstellung durch Zeichen. Ist uns ein Inhalt nicht direkt gegeben als das, was er ist, sondern nur indirekt durch Zeichen, die ihn eindeutig charakterisieren, dann haben wir von ihm statt einer eigentlichen eine symbolische Vorstellung. (Hua XII, 193)

Husserl's first extensive treatment of the logical problems posed by symbolic knowledge appeared in *Philosophie der Arithmetik*. A *symbolic* or *inauthentic* representation is a representation by means of signs, i.e., a content is not directly given to us but rather only indirectly through signs which univocally characterize it.[5] In fact, most of the numbers are given to us symbolically, thus Husserl proceeds to describe the way in which we perceive symbols and how they represent sets, collections or manifolds. Moreover, there are in fact two variants of this problem, one concerning the justification of the usual algorithms for carrying arithmetical computations, the other with treating the symbols 0 and 1 as proper numerical symbols. "One has to do with "blind" manipulations of meaningful symbols; the other with the use of meaningless symbols as if they had a meaning" (da Silva 2010, 127). In the first case, the algorithmic manipulation of numerals in the usual arithmetical operations is certainly not presided by accompanying intuitions; in that sense, "the symbolic system constituted by numerals and symbolic operations is an *isomorphic* copy of the system of number concepts and conceptual operations" (da Silva 2010, 127). In other words: if one has for numbers, for example, a set of basic principles, and it turns out that a set of basic principles formally coinciding, point by point, with the set of basic principles of arithmetic holds in an entirely different domain then is evident that corresponding to each possible arithmetical proposition is a proposition of the new domain and vice versa, in such a way that, as the basic principles, the inferences, conclusions, proofs and theories are also *isomorphic* (Hua XXIV, 84).

Immediately, Husserl realizes that there is a parallel structure of symbols and concepts. In other words, "Husserl's solution to the problem of extending the number domain by means of symbolic numbers relies on the idea of one-to-one correspondence between the signs (given in so-called normal form) and concepts" (Hartimo 2011, 153). So, calculation (*rechnen*) is a conceptual operation which utilizing the system of number signs, derives sign from sign according to fixed rules, only claiming the final result as the designation of a numerical concept (Hua XII, 257-258). With this definition of "calculation" we have obtained a true and proper characterization of the formal-algorithmic method. Hereby the notion of algorithm is bound up with that of a *mechanical process*. An algorithm is, in fact, a mechanical procedure that operates on configurations of (*sensuous*) signs according to certain formal rules. However, and this is very important, he did not think that operating with the symbols of a system according to prescribed rules constituted knowledge by and in itself. A calculus, he thought, although a useful technique, does not necessarily produce science. Finally, Husserl attributes great importance to this concept of calculation since it makes possible an exact separation of the various “logical” moments that are involved in every derivation of numerals from numerals. Undoubtedly the problem he was struggling with was the so-called *principle of permanence of formal laws*.

There is an important difference between a psychological and logical analysis. To Husserl, the symbolic synthesis and arithmetic analysis had been a constant source of difficulties, whereas no comparable difficulties emerged in dealing with smallest natural numbers. The distinction natural/formal manifested in arithmetic emerged and was developed to a higher level, to be considered not only as a methodological distinction but rather as an ontological distinction. In this sense, the reduction of the concept of number to a psychic act, and his collective connection, has failed, but it has made possible two positive things: (1) to accentuate the constitution (genesis) of the logical and mathematical concepts from the data of consciousness and (2) to separate the arithmetic technique from the conceptual domains making possible the application or extension of the arithmetic technique to any type of domains. To be clear, I am not saying that Husserl forgot and rejected the Brentano's empirical psychology and, in its place, placed another philosophy of mathematics; what I am saying is that he tried to assured the genesis of number in the acts and thinking in accordance with a set of rigorous scientific results to achieve an axiomatization of geometry which requires a principle that maintains the consistency of such extension and/or application in other numerical domains (Hartimo 2011). In other words, Husserl had realized that it should be possible to consider calculation entirely devoid of its conceptual basis. To Husserl, this means that instead of *extending the number domain*, one should rather talk about extending the *arithmetical technique*. To this new “project”, Husserl assumes the proposals of Hermann Hankel[6] and Hermann Grasssman,[7] on "Principle of the permanence of formal laws":

> Prinzip der Permanenz: Wenn vermöge der Besonderheit der einen Algorithmus begründenden Begriffe gewisse der algorithmischen Operationen nicht in voller Allgemeinheit ausführbar sind, ohne daß man auf widerstreitende Begriffsbildungen kommt, so erweitert man den Algorithmus, nachdem man ihn von der begrifflichen Grundlage losgelöst und als einen konventionellen gedacht hat, dadurch, daß man jede solche Bildung versuchsweise dem algorithmischen Gebiete adjungiert und die Konvention hinzufügt, daß auch für die durch sie symbolisierten Gegenstande (Zeichen) die alten Gesetze gültig bleiben, also die alten Gesetze unbeschränkt ausführbar sein sollen. Man muß dann in jedem Fall die Konsistenz des erweiterten Algorithmus nachweisen. (Hua XXI, 33)

In other words, according to Husserl's formulation, the principle allows extending the algorithm so that one can use the operations by stipulating that the old laws remain valid. Husserl emphasizes that the extended algorithm has to be shown to be consistent. Furthermore, Husserl investigates the correctness of algorithms by means of term reductions of an equational proof system. Indeed, "Husserl claims that the algorithms produce correct results when every equation for relations between the signs can be, using the definitions of the signs, reduced to an identity" (Hartimo-Okada 2016, 950):

> Wir erkannten schon, dass eine Arithmetik, welche die Zahlbegriffe zum Fundament hat, nicht etwa neben diesen noch andere Zahlformen, diesWort im eigentlichen und begrifflichen Sinn genommen, zulässt. Keine negativen, imaginären, gebrochenen Zahlen lassen sich nachweisen, die als Entwicklungsstufen oder Kombinationsformen der Anzahlbegriffe entstehen könnten. Der Anzahlbegriff lässt keinerlei Erweiterungen zu; was erweitert wird und Erweiterung zulä"t, ist nur die *arithmetische Technik*. (Husserl 1983, 42–43).[8]

The quotes suggest that Husserl realized that extending the numerical domain is only possible indirectly, i.e., by means of symbols. Thus, instead of extending the numerical domain, one should rather talk about extending the arithmetical technique. This "realization suggested him to detach the arithmetical technique from the conceptual domains and thus to allow the possibility of applying the arithmetical technique in any kinds of domains such as that of vectors. For Husserl this is justified by the *principle of permanence of formal laws*" (Hartimo 2011, 156).

The importance of this principle of permanence brought to Husserl the direct influence of H. Grassmann. The philosophical and friendly relationship between Husserl and Grassmann reveals little-known aspects of his philosophical developments, especially that which refers to the theory of manifold and to the principle of permanence of formal laws, of course. An example of this is Grassmann's little-known intervention in Husserl's attempts to generalize arithmetic beyond quantitative domains adopting a structural or purely abstract view of mathematics and logic. Equally unknown is the course that Husserl taught in the winter semester of 1889/90, where it is detailed how Grassmann, altogether with Gauss, represent an unprecedented break in the history of mathematics. Specifically, the description of the theory of parallels and the comparison of Gauss with Abel in terms of the theory of algebraic equations (Hua XXI, 318-322).

That course, actually, to belong to the manuscript K I 36. In unpublished pages of that manuscript, Husserl presents a summary of the introduction of the first edition of Grassmann's *Ausdehnungslehre*. Besides the *principle of permanence of formal laws*, Grassmann's general theory of forms has probably been the most influential aspect on Husserl's thinking and it can be seen as another effort in the line of attempts to combine a rigorous method of proof with a method that aids discovery. In this synopsis, according to Gerard (2010), is possible to observe three points that will define the theory of the "husserlian manifold".

> Seine wesentlichen Ideen, mit Ausnahme des Prinzips der Permanenz, verdankt er (nach seinem eigenen Zeugnis) Grassmann, vielleicht dem genialsten Mathematiker, den Deutschland in diesem Jahrhundert hervorgebracht hat. Bereits seiner im Jahr 1844 erschienenen ersten *Ausdehnungslehre* schickt er eine philosophische Einleitung voraus, in welcher er den Begriff der reinen Mathematik oder reinen Formenlehre aufstellt, welche das besondere Sein als ein durch Denken gewordenes auffasst. "Die Form in ihrer reinen Bedeutung, abstrahirt von allem realen Inhalte, ist eben nichts anderes, als die Denkform". (Manuscript K I 36, 8)

The *Formenlehre* of the *Ausdehnungslehre* of 1844, as the science of extensive and intensive types of connection, is identified with free mathematics. As such it governs all conceivable mental mathematical concepts. In that sense, the *Ausdehnungslehre* is recognized as the abstract foundation of the theory of space in which geometry is a particular application applied to space. Moreover, the general theory of forms describes a hierarchy of operations in terms of their relationship with each other. For example, the distributive property of multiplication over addition, from the right and left, is treated independently of the elements being added or multiplied. Thus, following Grassmann and Husserl, the *Ausdehnungslehre* is the abstract science dealing with methods of our outer intuition and hence it can be said to represent intermediary between transcendental philosophy and pure mathematics.

In the first paragraphs of the *Ausdehnungslehre*, Grassmann discusses the idea of an intellectual mathematics or general theory of forms as a redesign of the concept of pure mathematics. In this sense, Grassmann's general theory of forms is nothing but a set of "symmetry principles" valid in all of pure mathematics and expressed by means of equations. The calculus of extension as a particular mathematical theory results from an interaction of both the general science of forms and the intended applications which suggest important analogies to specify the axiomatic schemata.

So, this definition of pure mathematics results from the classification of sciences in real (*reale*) and formal (*formale*). The first science represents in thought the existent as standing independently over against thought, and have their truth in the correspondence of thought with that existent. The formal sciences, on the other hand, have as their object that which is posited through thought itself and have their truth in the correspondence of the reasoning processes among themselves. (Grassmann 1844, §1). The formal sciences consider either the general principles of thought or they consider the particular which is posited through thought—the former is dialectic (*logic*), the latter pure mathematics (Grassmann 1844, §2). The first is a philosophical science since it searches for the unity in all thought, while mathematics takes the opposite direction since it conceives each thought individually as a particular. According to Grassmann, pure mathematics is therefore the science of the particular existent as something which has come to be through thought. The "*particular existent that has come to be by an act of thought*" is a thought-form or, in short, a form. Thus, pure mathematics is the theory of forms (Grassmann 1844, §3). Each form is determined by its generating elements, which might be equal or different and by its generating act, either continuous or discrete. Forms are thus classified according to opposite concepts: discrete/continuous or equal/different. On the basis of this partition of forms in four kinds, which is dependent on their laws of generation, Grassmann classified mathematics in four branches: Number Theory, Theory of Intensive Magnitudes, Combinatorial Theory and Theory of Extensive Magnitudes (*Grössenlehre*), the latter not applicable to the theory of combinations and only improperly to arithmetic:

> Und durch eine abstr. philosophische Diskussion glaubt er die Spaltung der reinen Formenlehre in vier mathematische Disziplinen: in die Zahlenlehre und Kombinationslehre als die Wissenschaften der diskreten Form und die Lehre von den intensiven und die von extensiven Grössen als Wissenschaften der stetigen Form nachweisen zu können. (Ms K I 36, 8)

To understand the latter, it is necessary to make a distinction about the ways of generating forms. Grassmann says: "Jedes durch das Denken gewordebe kann auf zwiefache Weise geworden sein, entweder durch einen einfachen Akt des Erzeugens, oder durch einen zwiefachen Akt des Setzens un Verknüpfens." (Grassmann 1844, §4). The object generated in the first mode is continuous-form (*stetige Form*) or magnitude (*Grösse*); what is produced in the second way is the discrete-form (*diskrete Form*) or concatenated forms (*Verknüpfungs-form*). The intersection of the forms of generation results in the four main types of forms and, consequently, the four branches of the theory of forms. Correlatively, the sciences of the discrete are divided into number theory (arithmetic) and theory of combinations or theory of collective (*Verbindungslehre*) (Grassmann 1844, §6). The opposition between the two types of discrete forms is expressed in the unique sign that gathers the number, whereas the one gathered to form the combination is gathered in arbitrary letters. As for the continuous form or magnitude, it is divided into an algebraic continuous form (intensive magnitude) and the continuous combinatorial form (extensive magnitude).

Finally, the third point that Husserl quotes from Grassmann is the idea of a general theory of forms. With this point, Husserl concludes his exposition of the introduction of the *Ausdehnungslehre*, but he indicates that the four branches that constitute the theory of the forms must be preceded by a general theory of the forms, since "Diesen vier Disziplinen schickt er eine allgemeine Formenlehre voraus, welche die allgemeine, d. h. für alle Zweige gleich verwendbaren Verknüpfungsgesetze darstellt" (KI 36, 8). The general theory of forms would be concerned with establishing general laws of being insofar as thought develops itself. It is clear, according to Husserl, that such a theory does not yet exist; however, it is essential to theorize about it because it avoids the unnecessary repetition of the same laws in the four particular branches of the theory of forms and in its different sections. In accordance with this, Husserl retains three things from the philosophical introduction of the first *Ausdehnungslehre*: the definition of pure mathematics as the theory of forms of thought, the division of the theory of forms into four branches, and the idea of a general theory of the forms that will later take the name of the theory of magnitudes.

## 4. Theory of manifold in Husserl's "Early" Writings on Mathematics

Husserl remarks that a purely formal conception on sense of the "Object in general" was developed in the history of pure mathematics, in specific, with the systematic re-introduction of a line of thinking found on *methods of calculation*. Indeed, Husserl here invokes to Hankel and Grassmann on the one hand, as precursors of abstract algebra and "to Riemann on the other, as founder of the specifically so-called theory of manifolds generalizing on problems from (differential) geometry and analysis" (Cortois 1996, 43). So, "Object" here is no longer understood as a specific or determinate something to which we can refer arbitrarily, but rather as a "something" which itself has the sense of arbitrariness thanks to which it is uniquely accessible as a "general something" or a "general magnitude" for the method or procedure of investigation. This conception of Object in the Grassmann and Hankel's position (even the algebra of Vieta and Weierstrass) paves the way for understanding the "theory of arithmetic" as a pure construction of an explicit concept of a "something in general". Husserl reads this as the first emergence of the notion of what he calls a "domain undetermined" interpreted as a formal ontology, the laws of *mathesis universalis* would hold for all structures of meaning, as well as the mathematical manifolds, including the metamathematical manifolds of pure formal axiomatics.

Of course, Husserl still does not have the necessary clarity on the formal terrain and his steps are actually groping on the ground of an *arithmetica universalis* whose formal basis is the concept of manifold. Indeed, in his *Habilitationsschrift* and in the first part of the *Philosophie der Arithmetik*, Husserl maintains the idea that authentic numbers and the everyday conception of arithmetic based on simple operations justify the purely formal extension of it. However, when faced with the problem of the justification of the connection between authentic numbers (or authentic cardinalities) and formal (complex) numbers, Husserl moves towards a position in which numbers are determined according to formal systems. Though some real contribution from *Philosophie der Arithmetik*, where a general definition of manifold is attempted, might have been expected we have rather a general calculus of operations that Husserl identifies with a general arithmetic only after the publication of *Philosophie der Arithmetik*. Indeed, between 1893 and 1901,[9] Husserl considered that the world of formal axiomatics did not have to deal with real possibilities in order to articulate what is given within its sphere of concerns. In any case, the world of the mathematical and purely logical is a world of concepts, where truth is nothing other than analysis of laws. Husserl thinks that if concept of the domain (or field) numerical remain just as undetermined as the object of a concept, then we say any "object" whatever. *The only determining factor is the forms*:

> Ein so unbestimmt und in volliger Allgemeinheit gedachtes und nur durch Formen naher determiniertes Gebiet nennt der moderne Mathematiker eine Mannigfaltigkeit. Und das theoretische System der formalin Folgerungen nennt er die Theorie dieser Mannigfaltigkeit. Besser hiese es Mannigfaltigkeitsform, und dafür ist der korrelative Ausdruck natürlich Theorienform. (Hua XXIV, 86)

We must remember here that this latter, i.e. *Mannigfaltigkeittheorie*, has linkage with Cantor's set theory. Set theory is derived analytically from the concept of set (from a set essence) which is expressed in the relation between a set itself and its elements. The set-essences make it impossible for the members of the relation to be identical. It belongs essentially to the concept of set that no set can contain itself as an element without contradiction. For Husserl, it is part of the idea of set to be a unit (a whole) which comprises certain members as parts in such a way that it is something new that is first formed by them. It belongs essentially to the concept of whole that no whole can contain itself as a part. So, as a kind of whole, a set is subject to the rules governing wholes and parts that stipulate that a whole cannot, without contradiction, be its own part. However, the linkage that holds together the elements of a set (*Menge*) is not based exclusively on the act of collating (psychic). Therefore, Husserl said, in set theory, we make judgments universally about sets that in a certain way are higher-order objects. We do not make judgments directly about elements, but about whole totalities of elements and arbitrary elements, and the whole totalities, the sets to be precise, are the objects-about which. Here ends the possible influence of Cantor.

Instead, Husserl believes that the concept of number has to be something fundamentally different from the concept of collection, which was all that could result from reflecting on acts. Such doubts eventually undermined his confidence in the theories of Brentano, as well as those of Weierstrass and Cantor (Hill and Haddock 2000). Regarding this, Husserl discusses in the *Prolegomena*, how one can understand a modification of the mathematical concept of manifold; same concept that he learned as student in Berlin. It is certainly the case that Husserl adapted the concept of manifold to his philosophical and epistemological needs and took it beyond Riemann and Cantor, beyond mathematics and geometry, and into logic and formal ontology. Husserl is well aware of the differences between Cantor and Riemann. However, in most of his writings, when Husserl is discussing the notion of manifold, he has in mind Riemann, Helmholtz and Hankel and Grassmann as interlocutors, of course (Gauthier 2004). In that period, the notion of manifold was used by Husserl in almost exclusively technical contexts and, only, under a philosophical approach.[10]

Husserl developed his own conception of the theory of manifold that is even more general than the modern geometric or topological conception. Besides, he says that his conception was influenced by Grassmann's conception of an 'extension', Riemann's conception of a manifold. In Hua XXI this is more than clear:

> Cantor versteht unter Mannigfaltigkeit schlechthin einen Inbegriff Irgend geeinigter Elemente. Grundlagen einer allgemeinen Mannigfaltigkeitslehre, Leipzig 1883, S. 43, Anm. I: „Unter einer Mannigfaltigkeit oder Menge verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken läßt, d.h. jeden In begriff bestimmter Elemente, Welcher durch ein Gesetz zu einem Ganzen verbunden werden kann. (Hua XXI, 95)

In this sense, Husserl begins answering his question by noting that Cantor's *Mannigfaltigkeit* merely meant an aggregate of any elements combined into a whole. Husserl goes on to mention that Cantor's concept does not correspond to Riemann's and other related ones in the theory of geometry: "Was ist das, eine "Mannigfaltigkeit"? Zunächst nichts weiter als, in völliger Unbestimmtheit und Allgemeinheit gedacht, ein "Inbegriff" oder eine "Klasse" von Gegenständen. Nun, das sind doch lauter kategoriale Begriffe" (Hua XXIV, 88). Husserl stresses a *Mannigfaltigkeit* is not only an aggregate of elements that are just combined into a whole, but are ordered and continuously inter-dependent. Indeed, a *Mannigfaltigkeit* is not an aggregate of elements without relations. It is precisely the relations that are essential and serve to distinguish the manifold from a mere aggregate or set. Besides, for Husserl manifolds are not aggregates of elements without any relations. It is precisely the relations that are essential and serve to distinguish a manifold it from a mere aggregate (Hartimo 2011, 2016; Hill 2003). As explained above, Husserl saw manifolds as aggregates of elements that are not just combined into a whole, but are continuously interdependent and ordered so that each member possesses an unambiguous position in relation to any other one. The properties and relations set out in the axioms of a complete manifold in Husserl's sense determine objects unequivocally, bring information and logically eliminate nonsensical conclusions. Husserl said:

> Eine Mannigfaltigkeit ist nicht ein Inbegriff beziehungsloser Elemente. Gerade die Beizehungen sind das Wesentliche und Auszeichnende gegenüber einem bloßen Inbegriff. Nun liegt die Frage doch nahe: Welche systematische Form müssen die Beziehungen haben, welchen Charakter ihre einzelnen Elemente, damit ein System von Sätzen sich ergibt, das der Geometrie entspricht? (Hua XXI, 410)

Cantor used the terms *Menge*, *Mannigfaltigkeit* and *Inbegriff* interchangeably. Right from the beginning Husserl, however, directly confronted the terminological difficulties that one faces when speaking of *Mannigfaltigkeiten*. In his earliest writings, he noted that in place of the word *Vielheit* the practically synonymous terms *Mehrheit*, *Inbegriff*, *Aggregat*, *Sammlung*, *Menge*', etc. (variously translated by "quantity, aggregate, plurality, totality, collection, set, multiplicity") had been used. He acknowledged the ambiguity that comes with trying to define *Menge*. At the beginning of *Philosophy of Arithmetic*, Husserl informs readers that, while recognizing the differences, he would not initially restrict himself to using any one of these terms exclusively. By that time, the manifolds of the *Mannigfaltigkeitslehre* that Husserl himself was developing, and that he ultimately considered to be the highest expression of pure logic, were quite different from Cantor's *Mannigfaltigkeiten* (Hill and Haddock 2000, Chapters 7, 8, 9). From that time on, he strove to distinguish sets from manifolds, multiplicities, totalities, aggregates, etc.

But what exactly does the term manifolds mean? What is a manifold? According to Husserl, manifold is not nothing more than an "aggregate" or a "class" of objects conceived in complete indeterminacy and universality. Simple stated, the concept of manifold in its philosophical sense emphasizes its orderly form. A manifold is a structure, which is defined by its relationships. It is not defined by its own objects but by the set of values of a variable according to its parameters. A manifold is more than a collection of objects that are thought of as completely indeterminate; the manifold is a pure form with no other particular content than that its connections (or laws) that give it its validity. As such it is the basic concept of the theory of manifold. The theory of manifold is the investigation of the forms possible of objects domains as such; that is, when objects of thought have been cleared of the last remnants of intuitive content, which survived even in such notions as set or number, there still remains something to be said about the form of a domain of objects as it appears in all formal mathematical theories alike. In short, the theory of manifolds is for Husserl the theory of science itself. While it may belong, as task, to formal mathematics it is related to the elucidation of all possible forms that any scientific theory may take. So, when Husserl talks of logic or mathematics or ontology, he is referring to a theory of science, responsible for investigating the possible forms, or manifolds, that all deductive systems must adhere to. For Husserl then, a manifold is the form of an "infinite object-province" which can be unified under the exact laws of a nomological science. In terms of higher order, the notion of variety governs the form of a theory and defines its correlates in a relational way. Husserl insists that a variety is an aggregate of elements that do not combine in a whole, but are ordered and continuously interdependent.

Husserl was explicit that he borrowed his concept of manifolds from the contemporary geometry; now, sometimes Husserl also mentions in particular, besides Cantor's theory of sets, Lie's study of transformation groups, Grassmann's theory of extensions, and Hamilton's theory as similar attempts to capture the theory of all theories. In this sense he called the theory of manifolds *a fine flower of modern mathematics*.[11] This means that this new discipline, the theory of manifolds, was not only Husserl's vision. It turns reality in the last years of the nineteenth-century mathematics. Husserl's dream was to extrapolate this discipline to the whole categorical realm of human knowledge:

> Wenn ich oben von Mannigfaltigkeitslehren spreche, die aus Verallgemeinerungen der geometrischen Theorie erwachsen sind, so meine ich natürlich die Lehre von den *n*-dimensionalen, sei es Euklidschen, sei es nicht-Euklidschen Mannigfaltigkeiten, ferner Graßmanns Ausdehnungslehre und die verwandten, von allem Geometrischen leicht abzulösenden Theorien eines W. Rowan Hamilton u.a. Auch Lies Lehre von den Transformationsgruppen, G. Cantors Forschungen über Zahlen und Mannigfaltigkeiten gehören, neben vielen anderen, hierher. (Hua XVIII, 252)

As time goes, Husserl will relegate the problematic or purely mathematical approach to concentrate exclusively on the philosophical field. The establishment of this new course and the attention to logical-formal studies, will make the theory of manifold an essential edge of phenomenology through which various stages ranging from the understanding of complex numbers to pure logic. To minimize its importance is to belittle the philosophical work of Husserl himself. In short, Husserl will have to consider the theory of the manifold of modern mathematics as an embodiment of the ideal of a science of possible deductive systems, but which only partially represented the realization of his own ideal of a science of such deductive systems (Hill, 2003, pp. 173).

> Diese Andeutungen werden vielleicht etwas dunkel erscheinen. Daß es sich bei ihnen nicht um vage Phantasien, sondern um Konzeptionen von festem Gehalte handelt, beweist die „formale Mathematik" in allerallgemeinstem Sinne oder die Mannigfaltigkeitslehre, diese höchste Blüte der modernen Mathematik. In der Tat ist sie nichts anderes, als in korrelativer Umwendung eine partielle Realisierung des soeben entworfenen Ideals. (Hua XVIII, 250)

## 5. *Husserl's* idea of *a theory of manifolds* and Formalization

Husserl's theory of manifolds can be interpreted in three different ways (Milkov 2005): (i) his notion of manifolds was seen as being close to Riemann's theory of varieties; (ii) most often Husserl's concept of manifold was explained referring to the manifold of three dimensions in Euclidean geometry, and

(iii) Husserl followed the general theory of forms or polynomials by Leopold Kroneker's work foundations of an arithmetical theory of algebraic quantities (see Gauthier 2004). I will explore only two aspect (i) and (ii).

Husserl described manifolds as pure forms of possible theories which, like molds, remain totally undetermined as to their content, but to which thought must necessarily conform in order to be thought and known in a theoretical manner. So, we have a new discipline and a new method constituting a new kind of mathematics, the most universal one of all. Here formal logic deals with whole systems of propositions making up possible deductive theories. It is now a matter of theorizing about possible fields of knowledge conceived of in a general, undetermined way and purely and simply determined by the fact that they are in conformity with a theory having such a form, i.e., determined by the fact that its objects stand in certain relations that are themselves subject to certain fundamental laws of such and such determined form. In the previous sense, it becomes a theory of the form of theories whose objective is to investigate the essential concepts and laws inherent in an idea of science. It is also an investigation into the possible forms of object domains as such; that is, when the objects of thought have been eliminated from the last bits of intuitive content, which survived even in notions as a set or number (even in the formal sense), there is still something to be said about the form of such an object domain.

For all of the above reasons, Husserl discusses mathematics as a calculating technique, in specific, how the same technique of calculation can be applied in different domains, i.e., every concept in one domain corresponds to a concept in the other and vice versa or every operational concept corresponds to an operational concept in another domain. But, before is necessary known how the technique of calculation works or rather what is the procedure that follows the theory of manifold:

> In der Mannigfaltigkeitslehre ist z.B. + nicht das Zeichen der Zahlenaddition, sondern einer Verknüpfung überhaupt, für welche Gesetze der Form $a + b = b + a$, usw. gelten. Die Mannigfaltigkeit ist dadurch bestimmt, daß ihre Denkobjekte diese (und andere, damit

> als *a priori* verträglich nachzuweisenden) "Operationen" ermöglichen. (Hua XVIII, 251)

According Husserl, pure mathematics produces "calculation truths" of any kind. In other words, instead of numbers, energies, things, etc., He claims that it is better to think of letters and of rules of calculating. [12] In *Einleitung in die Logik und Erkenntnistheorie*, Husserl said that is incomparably easier to think of *a b c* only as something with which one is allowed to replace the form *a* + *b* or *a* * *b* or *a* – b, etc. Furthermore, letters and rules of calculation are enough. Letters and signs for connectives, it is easier to arrive at the combinations in general possible than with concepts (Hua XXIV, 84). If we accept this, then the problems in mathematics will be resolved in the higher possible completeness and generality. Rather, it is a "mathematics" of an indefinitely general realm of thinking. The only thing that is determined in it is the form. This approach was originally developed by Descartes' analytical geometry i.e., a science that solves geometrical problems reducing them to algebraic equations. In other words, is about to translates the intuitive properties of figures into a formal/algorithmic language that describes space within the quantitative frame of coordinates. Here emerges a connection between what is formal and what is analytical that will be further developed in his *Logische Untersuchungen* (and in *Ideen*). In these works, Husserl defines "formalization" (*Formalisierung*) as the procedure eliminating any material content from the proposition. In the end, we obtain a formal structure such that we can replace all material contents with an empty formal "whatever" without altering the logical form of the proposition. "Despite the early notion of "formal," it still overlaps the notion of "algorithmic" inherited by the *Philosophie der Arithmetik*, and despite the word "formalization" has not been coined yet, Husserl already conceives the first step towards a formal representation as an elimination of any material content" (Caracciolo 2015, 37-38).

The development of the notion of *formalization*, as a procedure of elimination of any material content, results in a formal structure that replaces all material contents with a mere formal void, that is, a *mere something in general* without altering its logical conformation (Caracciolo 2015). Due to the symbolization is mechanic, we can represent concepts through intuitions standing for them: for example, a real point may stand for the concept of point because watching the former we catch a symbolic link to the latter. This connection implies that intuitions and concepts are both different and similar in a way that Husserl does not further clarify. Furthermore, symbolization (whose content *is not* directly given to us) is defined as a mere negation of intuition (whose content *is* directly given to us), and therefore, its representational domain is reduced to what *is not* intuitive. As a consequence, symbolization has not an autonomous representational status.

Indeed, once one discovers that the deductions, series of deductions, continue to be significant and are valid when one assigns another meaning to the symbols, one is free to liberate the mathematical system, which can henceforth be considered as the mathematics of a domain in general, conceived in a general and indeterminate way. It is no longer restricted to operate in terms of a particular field of knowledge, we are free to reason completely on the level of pure forms. Operating within this sphere of pure forms, we can vary the systems in different ways. So, the idea of a theory of manifolds itself seems to draw mainly on the oldest of deductive-axiomatic disciplines. Ii is the "purification" of geometrical thought. By means of this method, Husserl said, people first became fully aware of the role of logical form compared to the content of knowledge, and as a further consequence a new discipline and methodology developed out of this that rose above all particular calculating disciplines and constituted a new mathematics of the most universal kind of all, a supramathematics, so to speak, a higher-level mathematics, a theory of theories as theory of possible theory forms (Hua XXIV, 84). To be more precise, when abstracting from the essentially material directedness of geometry, arithmetic or logic, some core element remains intact: the prototype "deductive theory as such".

About this last line, Husserl claims that we can discover these essential grounds of science by reflecting on *conditions* of science itself. For an investigation of these conditions one must look in two totally different directions.

First, we need to examine some objective logical laws that every science must obey in order to avoid nonsense or contradictions inside of their theoretical architecture (these logical conditions must be fulfilled by any science). In addition to this, we must examine the mental acts of knowledge in which scientific truths are given to us. This examination is directed towards the subjective conditions of knowledge, and it leads to an elucidation of the epistemological conditions of scientific knowledge and knowledge in general. Without such an elucidation of these conditions, Husserl believes that the sciences remain naïve, that is, without an understanding of their origin and essence. According to Husserl, every science is not just a collection of sentences about a certain field of knowledge, but rather a theoretical unity. Its sentences must be interconnected, because otherwise there would be no reason for us to call a mere collection of sentences a theory or a science. From a logical point of view, the unifying elements of this necessary interconnection between the sentences are certain logical laws and rules, e.g., the syllogistic inferences. Thus, the unity of science is based on the logical interconnection of sentences that is made possible by formal-logical rules. These formal structures are the theory-building elements in any science. This purely logical form of a theory can be investigated by logicians, because all these formal elements retain a certain independence from the concrete material content which they combine into a theory. Due to this independence of the logical form, it is possible to investigate all these theory-building elements in a general theory of science. The development of such a theory of science, that is, a theory of the formal structures of any theory, is, according to Husserl, the ultimate goal of theory of theories.

**NOTES**

[1] The invention of this notion is usually attributed to Riemann. In fact, the term "*Mannigfaltigkeit*", of which the word "manifold" is an English translation, appeared for the first time in the world of mathematics in Riemann's famous *Habilitationsvortrag*. There are other English translations such as "multiplicity" or "variety" in the mathematical literature. In this text

the choice has been made to follow the lead of David Carr, Dorion Cairns, Burt Hopkins and Dallas Willard.

[2] Only recently has it been discussed in a number of essays, Scanlon (1991), Majer (1997), Hill (1995, 2000), da Silva (2000, 2016), Gauthier (2004), Hartimo (2007), Centrone (2010, 2017), Okada (2013).

[3] In the preface to the *Philosophie der Arithmetik*, Husserl acknowledges the influence that Gauss's study on complex numbers exerted on him. Indeed, Gauss plays an important role in Husserl's mathematical formation: "Vielleicht erweckt es von vornherein kein ungünstiges Vorurteil für meine Bestrebungen, wenn ich sage, dass ich die Grundgedanken meiner neuen Theorie dem Studium der vielgelesenen und doch immer nur einseitig ausgenützten Gaussschen Anzeige über die biquadratischen Reste (II) verdanke" (Hua XII, 8). Also *cf.* (Hua XXI, 322–347). Gauss' work is important for Riemann and Husserl in two respects. First of all, it contains a systematic introduction of imaginary and complex numbers as an extension of the real numbers, and secondly, Gauss proceeded to substantiate these impossible numbers by providing a visual and geometrical characterization of them.

[4] Husserl's *Philosophy of Arithmetic* is a dialectical work. "It consists of two parts: the first part focuses on "psychological" investigations of the concepts *multiplicity*, *unity*, and *number*, insofar as they are given to us authentically and not indirectly with a mediation of symbols […] In the second part Husserl takes up the "logical" and "arithmetical" investigations" (Hartimo, 2011, p. 151).

[5] Besides, there are minor texts of that period published in Hua XII, XXI and XXI where Husserl's treatment presents basically three versions of the problem of symbolic knowledge.

[6] H. Hankel presented the principle of permanence in his *Theorie der complexen Zahlensysteme* (1867); Husserl knew and discussed Hankel's principle at least since his *Habilitation* in July 1887. Indeed, in order to habilitate Husserl defended eight theses in a disputation at University of Halle in 1887. One of the theses is "Das Hankelsche "Prinzip der Permanenz der formalen Gesetze" in der Arithmetik ist weder ein "metaphysisches" noch ein "hodegetisches" Prinzip" (Hua XII, 339). Even, Husserl had attended Hankel's lectures in the University of Leipzig, (*cf.* Schuhmann 1977, 4).

[7] In 1876–1878, before of his studies with Weierstrass and Kronecker in Berlin, Husserl studied mathematics, physics, astronomy, and philosophy at the University of Leipzig (Schuhmann 1977, 4). Of these years in Leipzig, date the friendship between Husserl and Hermann Grassmann's son and Robert Grassmann's nephew, Hermann Grassmann, Jr. It is not so well known that during the winter semester 1877-78, Husserl received the *Ausdehnungslehre* from Hermann E. Grassmann as a gift (Schuhmann 1977, 6) Also *cf.* Hartimo (2011) and Gérard (2010).

[8] Manuscript from around 1889–1890, quotes in (Hartimo 2016, 155).

[9] About the second volume (which was never published), Husserl wanted to provide or to communicate more details on investigations concerning to symbolic representations and the methods of cognition grounded on them.

Also, he wanted to show that arithmetic will appear as one member of a whole class of arithmetic, unified in virtue of the homogeneous character of identically the same algorithm.

[10] According to da Silva (2000).

[11] In *Ideen I*, Husserl said: "Mit anderen Worten, die Mannigfaltigkeit der Raumgestaltungen überhaupt hat eine merkwürdige logische Fundamentaleigenschaft, für die wir den Namen "definite" Mannigfaltigkeit oder "mathematische Mannigfaltigkeit im prägnanten Sinne" einführen. Sie ist dadurch charakterisiert, dass eine endliche Anzahl, gegebenenfalls aus dem Wesen des jeweiligen Gebietes zu schöpfender Begriffe und Sätze die Gesamtheit aller möglichen Gestaltungen des Gebietes in der Weise rein analytischer Notwendigkeit vollständig und eindeutig bestimmt, so dass also in ihm prinzipiell nichts mehr offen bleibt. Wir können dafür auch sagen: eine solche Mannigfaltigkeit habe die ausgezeichnete Eigenschaft mathematisch erschöpfend definierbar" zu sein. Die "Definition" liegt im System der axiomatischen Begriffe und Axiome, und das "mathematischerschopfende" darin, dass die definitorischen Behauptungen in Beziehung auf die Mannigfaltigkeit das denkbar größte Präjudiz implizieren -es bleibt nichts mehr unbestimmt" (Hua III/1, 153)

[12] According Husserl, the rules in general are given for operating with sums, products, quotients in arbitrary combination, etc. All these operations are used *as* mechanical rules of calculation. The letters are manipulated like a *game tokens*. Indeed, one can calculate with concepts and with propositions in the same way as with lines or surfaces. The calculation is not calculation with quantities and numbers, but only belongs being logically deduced. (Hua XXIV, 81-82).

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