

How Does Mathematics Get into Science, and Why? A Husserlian Perspective

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Abstract

In this paper, I present, first, a categorization of the many uses mathematics has in science as a methodological tool. I identify four: representative, instrumental, predictive and heuristic. I introduce the issue in a historical context, discussing it more systematically afterwards. My approach is Husserlian thoroughly, which means that I hold the following views: 1) real nature is perceptual nature, constituted out of the hyletic material of the senses by the action of built-in psychophysical proto-intentional systems; 2) mathematized nature is an intentional construct devised for methodological purposes; it instantiates idealizations of formal-abstract structures of perceptual nature, but can also incorporate non-representing (imaginary) elements; 3) mathematics serves science by offering contexts of representation of perceptual reality and instruments of theoretical investigation of mathematical substitutes of reality. I conclude by contrasting my approach with Husserl's own.

Keywords: Phenomenology, Husserl, mathematics in science, idealization, intentionality

A philosophy is alive when it inspires, when it offers instruments with which to think. Particularly about questions its creator may have tackled, but not exhaustively or with the same interest one may have. When a philosophy is good only for exegetical rumination, incapable of addressing present-day problems, it became a museum piece.

Husserl is a philosopher still very much alive. His philosophy offers a wide perspective from which to consider a variety of contemporary philosophical problems. The applicability of mathematics in physical science, which I will

tackle here, is one of them. My treatment of the problem will sometimes be very close to Husserl's own, but will sometimes diverge into domains that Husserl himself did not tread. However, I will never abandon a Husserlian perspective.

At the end of the paper I will point out what in my treatment can already be found in Husserl's published work and how I think one can fully explain and justify the applicability of mathematics in science (and how inefficient mathematics is as guide to metaphysics) from a perspective that is Husserlian in spirit but that Husserl himself did not completely embrace, not for narrowness of sight but to remain faithful to some basic tenets of his epistemology and the call to personal responsibility that lies at the center of his philosophy.

1. The scientific revolution of the 17th century was marked by two apparently conflicting tendencies; one, more accurate observation and experimentation, another, mathematization. But whereas observation and experimentation are modalities of perception, mathematization is the negation of it. To mathematize is to place physical reality beyond the possibility of perception, to make it perceptually inaccessible. There are then two essentially distinct conceptions of realities in modern science, the perceptual and the mathematical.

Which is *real* reality? How do they relate to one another? If reality is that which we can, at least in principle if not actually, perceive, what does mathematics, whose domains are not perceptually accessible, have to do with it? But if nature is, at its inner core, mathematical, what is the role of perception in empirical science, particularly in the validation of physical theories?¹ Which role does mathematics play in modern science and how did it come to play this role? Can it be logically-epistemologically justified?

Mathematics entered the domains of natural science cautiously in the beginning, with the application of geometry in the study of the motion of bodies in physical space, but progressively ever more intrusively to the point of becoming indispensable to and overwhelmingly present in modern science. The use of geometry as an instrument of representation and investigation of the kinematics and dynamics of bodies in

space and of algebra as a means of expressing and dealing with quantitative relations among physical magnitudes pose, apparently, no mystery: after all, bodies, the space and trajectories in space have geometrical properties, and evaluating and comparing quantities are practices firmly established in our pre-scientific life. But, obviously, neither geometrical figures nor mathematical numbers are objects of perception. There is a gap to be filled between, on the one side, perception and the world of our common practices and, on the other, mathematical entities and mathematical reasoning. Things get more complicated with the development of science, when mathematical methods and objects with no immediate correspondents in perceptual reality or our common practices became ever more important and mathematics acquired other uses than the purely representational.

But before attempting to understand and ultimately justify the many uses of mathematics in science one must identify them. I want to do this by following, if only superficially, the historical development of modern science, paying attention to those moments when mathematics conquered extra territory and extended its range of scientific applicability.

Identification will be followed by explanation and, ultimately, justification. Understanding how mathematics can in so many ways be useful in science and justifying mathematization from a logical, epistemological and methodological perspective are philosophical tasks to which Husserl has greatly contributed, although not to the extent I think he could.

His last published work, the influential *The Crisis of European Sciences and Transcendental Phenomenology* (henceforth *Crisis*), is basically a piece of propaganda for transcendental phenomenology as, among other things, the correct way of restoring meaning to fossilized scientific practices, mathematization particularly. By going back to the enthronization of mathematical methodology in natural science, as a genetic phenomenologist, not a historian, Husserl was able to detect how mathematization came to be, its goals and what he believed to be its limitations. He managed also to uncover

the many layers of intentional action that went into the establishment of the method but were, he thought, “forgotten” by tradition with the consequent endorsement of a wrong interpretation of both the concept of nature and the mathematical method of scientific investigation of nature. It is mainly this misinterpretation that Husserl criticizes, not the methodology per se, although some questions can be raised as to the extent to which he was comfortable with the full range of mathematical techniques in science.

Much has been published on *Crisis* and Husserl’s analyses of “Galilean” science but my goal here is not to contribute to this literature. I use Husserl to my purposes. I believe that naturalism, which holds that the role of science is to describe what she finds in nature without in any way contributing to the constitution of nature itself, coupled with a wrong interpretation of the concept of nature, as pointed out by Husserl, obliterates any honest attempt at understanding the many uses of mathematics in science. But although I find Husserl’s analyses of the intentional constitution of nature in modern science, together with his explanation and justification of some of the uses of mathematics as a scientific methodology, correct, I also find them incomplete. Ironically, however, it is Husserl himself who offers the key to understanding the method: purely mathematical techniques and materially meaningless symbolic manipulations can, as scientific strategies, be *logically and methodologically* justified in formal ontology.²

By investigating the formal-logical relations between formal theories and formal domains, formal ontology can tell us when truths of one domain or theory (for example, purely formal-mathematical extensions of mathematical representations of perceptual reality) are true in other domains and theories (for example, mathematical representations of perceptual reality themselves), thus safeguarding purely symbolic-mathematical means of scientific investigation.

But let’s first better identify the diverse ways in which mathematics can be used as a methodological tool in science by following, as I said, a historical line.

2. Classical geometry has always been, from its beginnings with Thales and Pythagoras to its mature developments with Euclid and beyond, a science of perfect forms and their mutual relations, things that are not exactly of this world.

In the Platonic interpretation, geometry deals with ideal archetypes inhabiting a world of their own that this world where we live can at best only imperfectly instantiate. We can ascend to this *topos ouranos* only through reason, not the senses. In the Aristotelian interpretation, on the contrary, the ideal forms of geometry are idealizations of actual or possible abstract aspects of this world that, however, as the idealization they are, are not of this world either. In this interpretation, to ascend to the geometrical realm, perception must be complemented with abstraction and idealization, the exactification of the perceivable. Geometry was by then already far removed from its origins in land surveying.³

Despite its astronomical applicability, geometry had in antiquity no place in the science of the real world of our direct perceptual experience. The shapes of the world were not supposed to be, strictly speaking, geometrical; the quantitative aspects of reality were not supposed to be, strictly speaking, arithmetical. Before the Galilean revolution, empirical reality was *perceptual* reality, not an idealized copy of it where arithmetic and geometry *proper* had a place. For the ancients, the perceptual world could be measured, and bodies had perceivable shapes, but *perceivable* measures and shapes were *not* supposed to be approximations to a perceptually inaccessible core of mathematical exactitude lying deep *in* the world itself.

This radically changed with the development of modern science by Galileo, Descartes and Newton, among lesser emblematic names. It all began with Galileo's geometrization of the perceptual world, not as a mere methodological device, but as the uncovering of a geometric reality *within* the world. Geometry entered natural science by means of a radical reshaping of the concept of nature, no longer what we perceive with our senses, but a perfectly geometrical reality buried deeply inside it, inaccessible to perception, only reason. *Nature*, real nature, *became transcendent* and immanent nature, only

an essentially imperfect image of it available to the senses. Disguising an invention as a discovery is the inaugural act of modern science. For Husserl, this was the origin of the “crisis” of science. By taking a *product* for a *given*, science alienated itself from the constituting presuppositions on which it sits, unaware of its true nature and the scope and range of validity of its methods, among which Galileo’s creation, mathematization.

Let’s take a closer look at Galileo’s procedures in geometrizing the natural world. Two operations are fundamental: abstraction and idealization. By abstraction I mean a refocusing of intentional consciousness that brings certain aspects of physical reality to attention in detriment of others, for example, shape instead of substance (form instead of matter) and arithmetic proportions instead of causal relations (at least in Galileo’s typically kinematic treatment of motion). By idealization I mean mathematical exactification, for example, taking the shape of physical bodies as proper geometrical forms, *instances* of geometrical ideas.

At a certain point in the Second Day of the *Dialogue Concerning the Two Chief World’s Systems* (Galilei 1970), Simplicius (the spokesman of scientific conservatism) criticizes Salviati (Galileo’s alter ego) for supposing that a sphere touches a plane in a single point. Spheres, Simplicius reasons, can be quite heavy and would deform a plane on which they are placed, thus escaping the idealized situation. In his answer, Salviati first recalls that in financial transactions one calculates with numbers independently of the matter coins are made of or the merchandise being sold or bought, and then explains that “[...] when the geometer philosopher wants to see concretely the effects proved abstractly, he must eliminate the interference of matter; if this is done, I assure you that things will be as exact as in arithmetical calculations [...]” (ibid., 265ss). This exemplifies both abstraction and idealization, abstracting form in detriment of matter and idealizing real physical spheres and planes as ideal geometrical spheres and planes.

Clearly, Salviati’s world is a mathematical world in the *proper* sense; however, he does not take geometrical idealization as an operation of *substitution* of reality by

irreality, of what does not exist for what does, but as an operation of *prospection* of reality itself, as the unveiling of reality's inner, most fundamental reality.

Galileo follows Euclid's *Elements* very closely. For example, in discussing free-fall, velocity is *not* defined as the ratio between the space some mobile travels and the time spent in the journey, for this would go against Greek principle that ratios only make sense between homogeneous magnitudes. In a sense, for Galileo, numbers are not yet pure (*ibid.*, 27). In discussing free-fall along inclined planes, Galileo states that two mobiles have the same speed when they travel equal spaces in equal times. Of course, one can define inequalities of velocities analogously. Notice that here Galileo reduces linear continuum physical magnitudes to *geometrical* line segments and deals arithmetically with ratios of line segments representing homogeneous magnitudes in conformity with Eudoxus' theory of proportions as presented in Book V of Euclid's *Elements*.

It is often said that pre-Galilean physics is essentially non-mathematical, and that experience and observation were not as important for the ancients as they were for Galileo and his followers. This is at best an exaggeration. From Thales, the half-mythical creator of philosophy and mathematics, to Galileo, science was a mixture of observation, sometimes very accurate observation that could also be of a quantitative nature, induction and explanation, often based on clever analogies. The physics of Aristotle, for example, although not mathematical in the same way of Newton's physics, is also concerned with quantities and quantitative relations. In *On the Heavens*, for example, Aristotle explicitly states a quantitative law relating the times two bodies take to cover the same distance in free fall: they are, supposedly, in the inverse relation of their weights; a body twice as heavy takes half the time. This is the law that Galileo claims, in the *Dialogues*, to have *empirically verified* to be false: bodies in free fall, he claims, cover the same distance in the same time independently of their weights if we do not take the resistance of the medium into consideration (a disproof of Aristotle's law by reduction is also provided).⁴

Regardless of whether he indeed verified, or *could* have verified this principle empirically, Galileo's criticism helped to establish the myth that Aristotle was not a good observer. The truth, of course, is that Aristotle's principle is approximately correct for many cases of bodies free-falling in viscous media, the cases that Aristotle probably observed with more attention.

There are, however, aspects of Galilean science that are completely strange to Aristotle. The concept of quantity of the latter is that of practical life and quantitative relations were certainly not supposed to be any exacter than those of ordinary mundane transactions. In Galileo, on the other hand, quantities are *exact*, represented by geometrical segments with which he could operate *geometrically*. Also, whereas Aristotle had only logic to derive the consequences of general principles, Galileo had geometry, a hugely more efficient instrument.

The object of Aristotle's science is the physical world as actually given to the senses, whereas that of Galileo is an *idealized version* of it where mathematics proper finds a place. Aristotle's quantitative relations are not, strictly speaking, numerical – nor could they have been – and are supposed to be valid in the perceptual world. Galileo's are numerical relations proper that are, however, strictly valid only in an idealized world.

Descartes made two major contribution to modern science. One was the "invention" of space or, better, the modern scientific conception of physical space. By establishing that the essence of physical bodies is their extension, not their matter, Descartes established that reality is ultimately abstract, thus providing the key for the identification of real physical space with geometrical space.

His second contribution was the creation of analytic geometry, where geometrical constructions can be carried out symbolically by algebraic means. Although not the first time that symbolic manipulations entered the realm of mathematics, for Descartes was preceded in it by the Italian algebraists of the Renaissance who invented "imaginary" numbers, the creation of analytic geometry opened the doors to the symbolic in science, mathematics *as well as* physics. This huge methodological step forward, however, is more clearly detectable in Newton.

The mathematical methods of Newton are substantially more sophisticated than those of Galileo. And this is already evident in the first proposition of his *Principia*: the areas, which revolving bodies describe by radii drawn to an immovable center of force do lie in the same immovable planes, and are proportional to the times in which they are described.

The proof of this assertion depends on Galileo's principle of decomposition of velocities (the motion of the body is the resultant of an inertial tangential motion and an accelerated centripetal motion), the geometrical theorem that triangles with equal bases and equal heights have equal areas independently of the size of the two other sides, and the mathematical novelty, the process that we would call today of "taking the limit", originally a creation of Archimedes. As physics is concerned, however, Newton's truly original contribution was that of force.

In our pre-scientific experience, there are at least two ways in which a body can move, by itself, due to some internal cause, or by being pushed, pulled or otherwise acted upon by another body that usually is in contact with it. The ancients attributed a "soul" to things that could move by themselves, people or animals, and did not seem to see any mystery in a body somehow inheriting the movement of another body that touches it. But there are situations in which inanimate bodies move without being touched by other bodies, for example, smoke that flies up and a rock that falls. Aristotle saw both types as natural motions driven by an internal disposition to regain a natural place, inertial motion as we would say today, involving final but no efficient causes.

But there is one unnatural type of motion without direct contact that the ancients knew well, magnetic-induced motion. Naturally, they tried to explain it in terms of causes that were familiar to them, direct action and souls. Aristotle says that to explain magnetism Thales attributed souls even to inanimate bodies ("all things are full of gods"). Others abandoned animistic for materialistic explanations like effluvia, emanations from the magnetic stone that somehow, by some clever mechanism, "pulled" the pieces of iron.

Inanimate bodies acting upon one another at a distance without any intervening mechanism was out of question. Aristotle criticizes both animistic and materialistic explanations then available for magnetism, thus running out of possibilities for explaining the phenomenon. Since it does not fit well into his theory of motion, which required direct contact for efficient causation, Aristotle passes in silence over the phenomenon of magnetic induced motion.

Newton's stroke of genius was to introduce a notion, that of force, as a completely general cause of non-inertial motion, that is, non-uniform motion in inertial frames. After guaranteeing in his first law of motion the existence of frames of reference in which bodies would be at rest or in uniform rectilinear motion if sufficiently far from other bodies (in empty space particularly) he introduces by a definition, the second law, the notion of force as *that which causes acceleration, whatever this may be*. In the words of Newton:

Def. IV: An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line.

This force consists in the action only; it remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its *vis inertiae* only.

Modern science contradicts Aristotle's theory of motion twice: Galileo's principle of inertia had already eliminated the need of constant action for the preservation of motion, Newton's concept of force eliminates the need of direct contact for changing the state of motion.

Now, and this is very relevant in the story I'm telling, Newton does not care to tell us *what* forces are and *how* they act, by internal disposition, contact with other bodies, or some intervening mechanism.

"For I here design only to give a *mathematical* notion of those forces, without considering their physical causes and seats (my emphasis)", he says. One does not know what forces *are*, we only know what they *do* in quantitative terms: forces "cause" acceleration depending on an intrinsic property of the bodies upon which they act, their "mass", which measures the

body's willingness to have their state of motion altered. In the very last scholium of the book, he says:

“But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called a hypothesis, and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy”.

Clearly, only what has *quantitative* expression has a place in Newton's non-speculative “experimental philosophy”. Only by abandoning the world of material beings and real causal relations and moving to a mathematically purified context where only quantitative relations mattered, Newton could introduce a purely mathematical notion of force with which to unveil the mathematically precise *mathematical* structure of the mechanism that makes the world go around.

There are causal explanations in Newton's science, but incomplete. We know that gravitational forces keep the world together, celestial bodies spinning around each other, and the seas in regular tidal motion, but we ignore what these forces are and how they act. We only know that forces impart acceleration, which allow us to find out the future and past trajectories of bodies and then, in principle, their precise position in space at any point in time, and this is enough.

In mathematized physical science, which Newton's mechanics exemplifies magnificently, only quantifiable magnitudes and quantitative relations matter. New notions are admissible provided they are quantifiable, and one knows how they relate quantitatively to familiar notions. Definitions take the form of algebraic identities telling how *definiendum* and *definiens* relate quantitatively to one another. Properties that are not at *prima facie* quantitative are admissible only if they can be given quantifiable representatives. For example, the principle of inertia states that in inertial frames rest and rectilinear uniform motion are “natural” states, which can only be altered by the action of “forces”. But bodies have a “natural” tendency to preserve their state of motion. *What* this tendency consists of is immaterial provided we know how to measure it. The degree of “laziness” of bodies, their unwillingness to alter

their state of motion, finds its quantifiable representative in the concept of inertial mass, which can be precisely expressed numerically in terms of some standard mass. The mass of the body increases with the force necessary to impart upon it a given acceleration.

Other notions such as momentum (linear and angular), work, energy, etc., follow the same pattern. A particularly important methodological strategy can now be introduced, namely, the search for *invariants*, magnitudes that are conserved, for instance, energy or momentum, thus allowing the establishment of principles that have both explicative and heuristic virtues. Centuries after Newton, the neutrino was conjectured to exist only to guarantee the conservation of energy and other relevant quantities in certain nuclear reactions.

The geometrization of motion became possible after we learned how the position and trajectory of bodies depended on kinematic and dynamic variables and how the former depended quantitatively on the latter. In its time, Newton's *Principia* was the best expression of this new knowledge, the magnificent synthesis of geometry, kinematics and dynamics heralding a new science: mathematical physics.

Geometry was by then almost synonymous with mathematics, but that was about to change. The analytic geometry of Descartes and Fermat, where geometrical constructions are replaced by algebraic manipulations, opened the doors to *symbolic reasoning* in mathematics. It would soon be adopted in physics too, proving to be in both domains an immensely rich and useful methodological strategy. By symbolic reasoning I mean reasoning by manipulating symbols according to rules without necessarily taking into consideration what the symbols mean or whether they have meaning (symbols usually stand for something, but not necessarily).

Despite its practical utility and logical justification, Husserl believed that this strategy presented risks. Symbolic reasoning, he thought, cuts us from the things themselves, we are no longer directly concerned with intuitive contents, only with intuitively empty symbols that, in the best possible case, stand for things that can only *in principle* be intuited. By

forsaking intuitiveness, that is, perceivability, science became mechanical and ritualized. This was, Husserl thought, the beginning of the crisis of science that would eventually develop into a crisis of culture in general characterized by loss of meaning and direct personal responsibility.

The problem gets considerably worse if symbols exist in the symbolic machinery of science that have no representational value. In this case, intuitiveness not only recedes into the background but is completely lost. Here a tension appears between undesirable symbolic alienation and desirable methodological effectiveness. Meaningless symbolic manipulation had at least since the 16th century proven its value as a methodological mathematical tool with the introduction of “imaginary” numbers in the theory of algebraic equations. There was no a priori reason why symbolic reasoning should not be methodologically relevant in mathematical physics as well. Indeed, time has shown that this strategy is not only useful but central in scientific practice. Although non-representing symbols have no *representational* value, they can still play *instrumental*, *predictive* and *heuristic* roles in science.

This considered, is Husserl’s criticism of “symbolic alienation” still relevant or should it be dismissed as old-fashioned and misguided conservatism? I believe Husserl’s cautious admonitions still resonate, although not as a critique of scientific methodology itself, which he never meant them to be, but of the absurd interpretation given to it. Husserl urges us to uncover, recover and vivify the sedimented meaning of our scientific practices. By tracing the intentional genesis of both the modern conception of physical reality and the methodological strategies devised for investigating it we can shun absurd accounts of the adequacy of these methods. As we will see, Husserl’s philosophy itself opens a possible route for explaining the efficacy of mathematization in science, even when involving meaningless symbolic reasoning, and justifying it on logical and epistemological grounds.

There are essentially two ways of accounting for physical action, action at a distance and action through a medium that “transmits” the action. Classical Newtonian theory of

gravitation favored action at a distance. Classical electromagnetism, with Faraday and Maxwell, preferred action through a medium, whose nature, however, changed dramatically throughout history, from something *real* to something *mathematical*. At first, electromagnetic action was supposed to be transmitted through a *material* medium that supported the electromagnetic field, the never detected ether, an elusive substance known only “indirectly” through its quantifiable properties. Later, the medium became space itself, but with the property of acting and reacting to the presence of electric charges and currents. More recently, with quantum field theory, the role of carriers of electromagnetic action was taken by photons. In relativistic theory of gravitation, at least until it is properly quantized, the medium is still space, or rather spacetime, whose *geometry*, however, depends on the distribution of matter. Bodies act on the geometry of space, which in turn determines how bodies move. This relation between matter and geometry, the world and mathematics is illustrated vividly in Einstein’s field equation, mathematical entities on one side of the equation representing physical entities, those on the other standing for geometrical entities. More than a formal *equivalent* of gravitational action, the geometry of space-time *is* gravity and *explains* its action. As we see, besides mathematizing reality, mathematization also often reifies mathematics. Mathematics does not simply *represent* reality, mathematics is *part of* reality. Some go as far as saying that reality is *nothing but* mathematics.

Mathematics is most active in quantum mechanics, playing therein all the roles it can play.

De Broglie conjectured that electrons and similar particles that carry energy E and momentum p are associated with waves with frequency ν and length λ such that $E = h\nu$ and $p = h/\lambda$ (introducing the wave number $k = 2\pi/\lambda$ and the angular frequency $\omega = 2\pi\nu$, $E = (h/2\pi)\omega = \hbar\omega$ and $p = \hbar k$). Here, mathematics is a *language* where to express physical correlations in idealized form, playing thus a *representational* role.

Schrödinger imposed upon himself the task of finding the equation of such a wave (Schrödinger 1926). It is not very clear

how precisely he did it, the heuristic strategy he used, but for my purposes this is not important, the following “derivation” suffices. Consider first a particle with definite energy; its associated wave has, then, definite frequency and can be represented as a sinusoidal wave: $\psi(t) = \sin(\omega t)$. Helmholtz’s equation applies: $\Delta \psi + k^2 \psi = 0$, where Δ is the Laplacian operator. Substituting one gets $\Delta \psi + (p/\hbar)^2 \psi = 0$. But $p^2 = 2mK$, where m is the mass of the particle and K its kinetic energy. Therefore, $\hbar^2/2m \Delta \psi + (E - V) \cdot \psi = 0$, where E is the total energy and V the potential energy of the particle. This was the original Schrödinger equation, usually written thus: $-\hbar^2/2m \Delta \psi + V\psi = E\psi$. (1)

Consider now the evolution of the state function of the particle. *Supposing* that $\psi(t)$ is a *complete* characterization of the state of the system at time t , the evolution equation must involve *only* the *first derivative* of ψ . But, under this hypothesis, ψ cannot be a sinusoidal function, it must have the more general form $\psi(t) = \exp(i\omega t)$. From this we get: $i\hbar \psi' = E \psi = -\hbar^2/2m \Delta \psi + V \cdot \psi$, which is the time dependent Schrödinger equation. Here, we see, first, how mathematics succeeds in providing an adequate context of *representation* – complex analyses – where both the form of the wave (*given the formal restriction* that the expression for the wave at one instant must determine its expression at any future instant) and *the presupposition itself* (the wave equation must be of first order in the time variable) can be expressed. And, second, how by using mathematics *instrumentally*, i.e. as a *context of derivation*, one succeeds in writing the evolution equation, the time dependent Schrödinger equation. Now, still using mathematics instrumentally one can, ideally, solve the equation and obtain the function $\psi(t)$ from which one can derive knowledge about the system that allows us to make predictions about it. Here, mathematics plays its *predictive* role (of course, these roles are not independent one of the others). The success or failure of these predictions determine the success or failure of the whole theoretical schema.

Historically, the development of quantum mechanics was heavily conditioned by the hydrogen atom (one proton and one electron) and its spectrum, characterized by lines of emission of radiation with different well-defined frequencies. If

a wave function ψ is supposed to characterize the state of the H atom, it is natural to see its states of definite energy as eigenvalues of an equation of the type $\hat{E}\psi = E\psi$ (2), where \hat{E} is some Hermitian “energy operator”, with state-functions associated with the states of definite energy as corresponding eigenvectors. ψ must, then, be a vector in a complex vector space. The formalism of linear algebra is then summoned as the adequate to express quantum mechanics (matrix mechanics).

Now, comparing (1) and (2) we see that $\hat{E} = (-\hbar^2/2m \Delta + V)$ and, therefore, the operator associated with the x-component of momentum must be $= -i\hbar\partial p/\partial x$, etc. Again, as we see, complex numbers cannot be avoided. This gives us a “recipe” of how to write the Schrödinger equation for a quantum system: write the Hamiltonian, “quantize” using the correspondence above and substitute in (2).

The solutions of the corresponding equation for the H atom fit well the experimental data, i.e. the known frequencies of spectral lines, showing the correction of the approach to the problem. Here, again, mathematics plays its predictive role and *because* it plays this role the entire symbolic apparatus can be put to test.

Notice that not all terms of the mathematical language plays a representational role. The symbols i and ψ , or those for the operators, for example, are not denotative, there is nothing in perceptual experience that corresponds to them directly. The situation is like that of language, in which some terms, such as names, denote but others, such as prepositions or conjunctions, do not, being only elements of internal articulation of the discourse.

Let’s consider now a particularly relevant instance of the *heuristic* role of mathematics in quantum mechanics, the mathematical “prediction” of the positron and antimatter in general.

By substituting the energy and momentum operators in the relativistic equation for the energy of a free particle, Oscar Klein and Walter Gordon succeeded in obtaining, in 1928, a relativistic version of Schrödinger’s equation (the Klein-Gordon equation). The equation can be generalized to particles under

the action of a potential. However, it did not agree to content with the experimental data when applied to the hydrogen atom. A characteristic feature of Klein-Gordon equation is that it is of second-order, the momentum operators are squared. Dirac considered that if the wave function provided indeed a complete characterization of a quantum system, then its value at some instant should be sufficient to determine its behavior in the future, and then the wave equation should be of first-order in the *time* variable. Now, he argued, since time and space variables are symmetric in relativity theory, the wave equation must be of first-order in *all* four spacetime variables. Based on these *formal considerations*, that the true quantum-relativistic equation of a particle must be of first-order on all variables, he arrived at a wave equation whose solutions, the wave functions characterizing the particle, are four-component spinors, each component obeying the Klein-Gordon equation.

Again, the formal restrictions Dirac imposed on his equation were the mathematical translation of one *desideratum*, namely, that the wave function should contain a complete characterization of the system, and the established physical *fact* that space and time are *formally* symmetrical in relativity theory. Mathematics functions here as a context of representation where both the desideratum and the physical fact are expressible.

Now, in accordance with Dirac's equation, the wave function of, say, an electron, has four components, each characterizing a possible state of the particle. Two of them, corresponding to states of positive energy, have natural interpretations, corresponding to two different possibilities for the spin of the electron, but the two remaining components, corresponding to states of *negative* energy, had by then no available physical interpretation.

Dirac had *two* alternatives, either to dismiss the negative-energy solutions as senseless mathematical sub-products of the formalism, or, more interestingly, look for some physical interpretation for them. But we should be careful here, there is nothing in the formalism itself pressing for the latter alternative and, even more importantly, nothing indicating *what* these physical things could be. Dirac was completely free to *guess*

what the “imaginary” solutions corresponded to, *provided* it had the required *formal* properties. Dirac guessed that these things were undetected electrons forming a “sea” of electrons (Dirac sea) trapped in all possible states of negative energy.

Now, once one of these electrons moved to a state of positive energy, it would leave behind a “hole” that would behave *formally* as a positively charged electron with positive energy. This is how far mathematics can go, *whether* there is something in physical reality corresponding to this “hole” and *what* it is, mathematics is completely silent about. Dirac, however, was free to conjecture that there *may* exist in nature particles just like the electron, but positively charged, whose states are given by the two components of the spinor that corresponded formerly to states of negative energy of the electron. A few years later (1932), Carl Anderson discovered the positive electron, the positron, showing that Dirac had *guessed* right. *But that remained a happy guess, not a prediction.* Of course, the guessing was from the start subjected to *formal constraints*, which is *all that mathematics can provide as a heuristic instrument.* To claim that Anderson’s particle is Dirac’s “hole” is to claim more than what the facts allow. Anderson’s positron has only the *formal* properties of Dirac’s “hole”; there was no a priori guarantee that these “holes” were real nor that they would manifest themselves as positrons, since they *could* in principle materialize as *anything* with the right formal properties. *In his heuristic role, mathematics unveils formal possibilities.* It is all it can do, but it is already a lot.⁵

3. This *vol d’oiseau* over the history of science illustrates what I believe to be the main uses of mathematics in science, representational, instrumental, predictive and heuristic. Summarizing:

1) *Representational.* In this role, mathematics offers contexts where certain formal-abstract aspects of physical reality are instantiated in an idealized manner.⁶ Mathematics represents to the extent that it provides contexts of instantiation (materialization) of idealized formal-abstract (structural) aspects of physical reality.⁷ One is often interested in mathematically representing only restrict structural aspects

of reality, not the whole of it, and in general in such ways that not all mathematical entities and situations in the context of representation are themselves representational.⁸

The single most important strategy in the process of mathematizing perceptual reality⁹ is *quantification*. The world is never the same, it changes in many ways; to quantify is to express the *quantitative* variability of the world as mathematical variables over numerical domains. For example, bodies have different volumes and the volume of a body may change in time (by volume one can simply understand the amount of space delimited by the body's surface). We can compare volumes perceptually and convince ourselves that bodies have always more, less or the same volume as other bodies. These are perceptual facts (effectively perceived or potentially perceivable), but perceptual quantitative relations are not yet mathematical; at best, they are proto-mathematical.

Quantifying the notion of volume amounts to expressing in numerical terms, in principle if not actually, the volume of *any given body* in terms of the volume of a standard body taken as reference, the unit. A *magnitude* is any physical entity that can be quantified, for instance, the volume of bodies; mathematically, a magnitude is a numerical variable ranging over the domain of all its possible values; i.e. all the numerical values representing the quantitative variability of the entity in question with respect to the relevant unit.¹⁰

Magnitudes can have determinations beyond the purely quantitative, such as direction in space (velocity, acceleration, and forces, for example), or be combined in mathematical entities more complex than pure numbers (e.g. tensors). There are no limits to how mathematics can build complex entities from numbers to serve representational and instrumental purposes (see next section), complex numbers, numerical functions of many variables, vectors, tensors, fields, etc.

It is important to emphasize that numbers, insofar as they represent quantitative relations¹¹, do *not* express them *as they are or can be experienced*; numbers express only *non-experienciable idealizations* of quantitative relations as they are perceived. The more and the less of perceptual reality can only be given a number by being idealized beyond the possibility of

perception. In short, to quantify, i.e. to express quantitative relations *numerically*, is already to idealize.

Now, when sufficiently many physical magnitudes are quantified, for example, volume and temperature, one can express mathematically by formulas, i.e. algebraic correlations among the variables standing for these magnitudes, how one perceives (or conjectures) they to be related; how, for example, volume changes with the change of temperature. But importantly, formulas not only express what is effectively perceived, but also what is perceivable but not yet perceived, thus offering a sort of *anticipation of perceptual experience* (allowing *predictions* to be made).

Structural aspects of physical reality other than the quantitative as, for example, spatial structure, can be mathematically represented by *substituting* perceptual space by an *isomorphic* numerical copy where spatial properties are expressible by numerical functions. We first label the points of space with n -tuples of numbers, their *coordinates* (n being the dimension of the space). This numerical labeling is not completely arbitrary for it must express the topological continuity of space (in Riemannian contexts) or its metric (in less general Euclidean contexts). Although the labels themselves do not express quantity, geometrical properties of space such as metric or curvature can be represented by numerical functions of the coordinates. The symmetries of perceptual space can also be represented by mathematical constructs in the numerical domain representing it.

General principles and laws that we believe (perceive, conjecture) to rule over the behavior of reality can also be given mathematical expression. For example, the law of universal gravitation or principles of invariance such as the principle of conservation of energy or angular momentum (infinitesimal calculus offers the ideal context where to express these and other conservation principles). There are many such principles in physics, playing predictive, explicative, and heuristic roles, conservation of energy being probably the most important. Variational principles such as the principle of least action, for example, are also central: a certain function of given physical magnitudes, the action, is supposed to be always either

minimal or maximal in physically real processes. Again, only in calculus such a principle can be adequately expressed.

Mathematical formulas, principles and laws express *mathematical* facts, representing in mathematically idealized form relations among physical entities and regularities in physical reality that cannot be adequately represented in any other way. Mathematical avatars of physical reality are not, even in principle, experienciable, but are assumed to be in principle perceptually approachable to any given degree of accuracy. Instead of a photographic copy of reality, mathematics provides an X-ray that captures only structure, but with an infinite degree of precision. Precision, however, that is not in reality itself, only in the way the X-ray machine operates.

2) *Instrumental*. The first and most crucial step in the mathematization of physical (perceptual) reality is the representation of certain of its structural aspects mathematically, i.e. *as* aspects of convenient mathematical manifolds that *substitute* reality. Once these manifolds are in place, their *mathematical* theories can be developed. Now, mathematics takes the lead; its task, to investigate by mathematical means the *mathematical representatives* of (formal-abstract aspects of) the physical world, to bring to light subjacent relations, organizing principles, hitherto unperceived correlations, in short, any structural aspect of reality representable in the mathematical context in question. Thus, mathematics becomes *instrumental*.

As such, mathematics is free to introduce terms that may or may not have representational value, provided they play a role in the internal organization of the theory. A good example is Schrödinger's wave function, not itself representative of anything real (only the square of its module, a real-valued function, represents something "real", a density of probability distribution).

Additional terms may, of course, also represent, but not necessarily. The scientist must always be alert to the possibility of "imaginary" terms representing *something* in physical reality, in which case mathematics plays a *heuristic* role. When mathematical manipulations disclose hitherto unknown facts

involving only *representing* terms, mathematics plays a *predictive* role.

3) *Predictive*. Mathematics predicts when it discloses mathematical facts that *must* correspond to facts in represented reality *if* the mathematical theory in question is a reliable representation of reality. If, for instance, the formula relating variation of volume to variation of temperature gives, after the required algebraic manipulations, a volume corresponding to a given temperature, then, if the volume is measured at that temperature and the formula is correct, one *must* get that volume.

Therefore, the fact that predictions can in principle be falsified by observable facts opens a road for theories to be empirically tested. Of course, no test is definitive; theories are always pouring out predictions that must be put to test. If the prediction proves to be correct, the theory passes the test. But tests never end, and final confirmation is forever postponed. Empirical verification can falsify a theory but never verify it once and for all.

Once a prediction is proven incorrect, the whole theory, its presuppositions, even its mathematical and logical setting is under threat. Fixing it requires ingenuity and is usually done in the most conservative way. There are no predetermined rules.

Now, something interesting can happen. As said before, to play its instrumental role to content, mathematics can introduce any additional terms it finds necessary to develop the theory. There will, of course, be “predictions” involving terms that have no correspondent in represented reality. Of course, these are not really predictions but, rather, meaningless sub-products of the mathematical machinery. They, however, allow mathematics to play maybe its most puzzling role, the heuristic.

4) *Heuristic*. Mathematics plays a heuristic role when it suggests that there *may* be things in physical reality, though *not for sure*, that regardless of what they are, *mathematics cannot say*, have certain formal-structural properties that the mathematical formalism seems to reveal, *but that could also be meaningless non-sense*.

There are diverse ways in which mathematics can help us in the risky business of guessing how reality may be like. Non-denoting terms that play only an instrumental role in the

formalism may correspond to hitherto unknown physical entities. Their efficacy in the derivation of observable consequences may be an indication of their physical reality. Hidden aspects of reality may announce their existence, *formally* if not materially, by the way they relate *formally* to known reality.

Formal “predictions” can also be derived from principles or presuppositions built into the formalism. For example, the possible existence of the neutrino or conduction currents conjectured as consequences of the principles of conservation of, respectively, energy and electric charge. Or the “prediction” of anti-matter. The *formal* properties that Dirac imposed on his quantum-relativistic equation expressed both necessary relativistic constraints and a well-established methodology of quantization. The fact that his formalism could naturally account for the spin of electrons indicated that spin is essentially a relativistic property. That it also opened a *formal possibility* for the existence of positrons showed that *something formally analogous* to positrons might also be a necessary consequence of the junction of these two desiderata, relativistic equivalence of space and time (and Lorentz invariance) and standard quantization techniques (quantum equation derived from the expression for energy by substituting standard operators for variables). In short, the “prediction” of anti-matter did not come out of the blue sky merely as a sub-product of meaningless mathematical manipulations; symbolic manipulations were only an *instrument* for deriving the necessary formal consequences of the formal expressions of *physical presuppositions* and a *methodological orthodoxy*.

4. The centrality of the representational role of mathematics in science requires closer attention: how and in what sense can mathematics represent reality? How can a mathematical world emerge from the perceptual world? Husserl in §9 of *Crisis* provides I believe, the answer: a mathematical world is constituted from the perceptual world by intentional action; the former represents the latter by replacing it as the object of science.

The life world: a world imposes itself upon us that we must either understand and control or die. The outside world has many dimensions, the physical is one of them. Our access to the physical world is essentially sensorial-perceptual; the sensorial is the purely material hyletic, the perceptual, the sensorial matter endowed with form, either given directly with the sensations or imposed upon them by the perceptual system itself: to perceive is already to constitute, at a pre-logical, pre-categorical, proto-intentional level.

The perceptual world is the physical world of our pre-scientific life, the physical world of the lifeworld; it is a real, concrete, materially filled, finite world. Our knowledge of the perceptual physical world is basically inductive, the result of non-scientific perception-based attempts at understanding and disclosing patterns and regularities in it that allow us to control it to some extent and survive. Science, on the other hand, is a rational endeavor whose most basic task is to improve our capacity of making inferences about the perceptual world¹², but that is not itself a practice of the lifeworld. Mathematization is a *further* methodological development of scientific practices.

The perceptual world is made of objects, processes, properties, correlations that display proto-mathematical (not yet properly mathematical) aspects. Bodies have form, although not geometrical form; they occupy position in space and maintain with one another spatial and kinematic relations that, however, are not properly speaking geometrical or chronometrical. Bodies have intensive and extensive properties such as height, length, volume and size (extensive), temperature, color and hardness (intensive) that can be measured by means available in the lifeworld, meters, balances and thermometers, but not with mathematical precision. Bodies can be compared as to the hue, luminosity and intensity of their color or their degree of hardness, but with some degree of fluidity and subjectivity. In the life-world, there always is room for arguing about which body is redder or harder. Bodies act upon one another, but not by means of precisely quantifiable mathematical agents.

Quantitative determinations of the perceptual world are, from a mathematical perspective, essentially imprecise, but

cannot be otherwise as far as perception is the privileged means of access to reality and objectivity is only another name for intersubjectivity. Mathematization is an attempt at overcoming subjective perspectives and perceptual “imprecisions”. However, since *only* the formal-abstract aspects of perceptual reality can be a matter of intersubjective agreement, for *form* only is *objectively real*, the cost of objectivity is the loss of essentially private perceptual materiality. The materially empty skeleton of perceptual reality can then be mathematically rectified, i.e. exactified, so mathematics can come in with full force as a methodological device.

The perceptual world, a dimension of the pre-scientific lifeworld where we live our daily lives, as is obvious, is not merely *given*, we do not stumble upon it readymade. Although there is, of course, a given of the world, it must interact with our perceptual systems for a perceptual world to be *constituted*. The perceptual world is a *product*, although not of a conscious and fully intentional process; it is the given of the world endowed with *perceptual sense*. To perceive is to make sense of the senses; perception is already a cognitive process of a fully active, although not fully conscious subject. The proto-mathematical aspects of the perceptual world are then as much a given as a contribution of the perceiving subject, not because it so wants, but because it cannot help it; the perceiver cannot choose to perceive differently from how it perceives because it cannot choose to be different from how it is. The way *we* perceive the world tells as much about the world as about us. The proto-mathematical aspects of the world are not necessarily out there; they can be, at least in part, a contribution of ours.

Our daily lives are directed to essentially practical ends, our own survival and the survival of our species being the supreme good. A certain capability of understanding, controlling and predicting the behavior of the world as perceived is an important instrument of survival to which certain practices of the lifeworld contribute, such as measuring, counting or comparing. This is not yet science but science will gradually emerge from these practices as a way of improving our ability to understand, predict and act on the world.¹³ Mathematization is only its most radical development, one that

goes beyond the perceptual world to better investigate *some* of its aspects – its proto-mathematical aspects, in particular – those precisely that can be mathematically idealized.

The mathematical-physical world. The mathematical-physical world is abstract (ontologically dependent), idealized (exactified), formal (materially empty), non-perceivable, and infinite; our knowledge of it is mathematical and deductive, based on theories, principles, and laws (for they only can theoretically master an infinite domain¹⁴).

The mathematical-physical world is *not* an independently existing entity, but a higher-level *intentional construct*; it is the mathematization of the perceptual world, requiring for its constitution many layers of intentional action. Abstraction or, more precisely, formal abstraction is one of them. By stripping the perceptual world of its material, intuitive, private content, it reduces it to its objectifiable abstract *form*. Spatial shapes regardless of material support and quantitative relations regardless of what is related, for example.¹⁵ Quantification is an instance of formal abstraction; once a magnitude is quantified it is reduced to a numerical variable. The mutual dependence of magnitudes in the perceptual world is, when mathematized, restricted to their quantitative aspects, expressible by numerical correlations, formulas and equations.

The spatial form of bodies, despite abstract, are still *real* forms in perceptual space. Their geometrization requires a further step, idealization, by which real forms are *exactified* as geometrical forms proper. To idealize something (a spatial form, a quantitative relation) is to take it as an instance of an idea (a geometrical form, a number).¹⁶ By so doing one can investigate the spatial properties of the real, perceptual world geometrically. Let's consider an example. Suppose one wants to know whether a rigid object of the physical world, say a right-hand glove, can occupy a predetermined place in space, say, that of a similar left-hand glove, by simply moving in space without changing its form. Since the problem involves only the spatial form, not the material content of gloves, we can consider it abstractly and, by idealization, geometrically: are the right-hand glove *form* and the left-hand glove *form* equal under rigid

motions in space? The *geometrical* problem can now be tackled and the negative answer obtained. The standard approach is *algebraic*, in terms of groups, because the problem has more to do with group-theoretical properties of motions in space than space itself. This only solves the initial *real* problem because abstraction and idealization can somehow be undone. Abstraction does not do away with matter altogether, it only puts it in the background as a standard semantics for filling (giving material content to) real forms mathematically idealized. By going back from mathematical to real forms, in the direction opposite of mathematization, and filling these real forms with their original material content, one gets from the mathematical a *physical impossibility*. By being purely formal, the impossibility of the body in question being placed in the place in question can be disclosed by a formal-mathematical investigation.

Another crucial step in the process of mathematization is the sorting out of qualities into primary and secondary. Primary qualities are objective and mathematizable; secondary qualities are subjective and intrinsically perceptual. Today, we take as an evidence that qualities such as taste, color and texture are secondary, residing in consciousness rather than in the object, but that spatial form is primary, residing in the object itself. But every evidence one may adduce for the subjective character of, say, color (dependence on subjective states, conditions of observation and the like), holds good for spatial form too. The privilege of primary properties is based on its willingness to be mathematized. If, say, color, could be as easily quantified in terms of *objective* standards of hue, saturation and brightness there would no doubt exist a mathematical theory of colors independently of electromagnetic theory (where color is reduced to a quantifiable magnitude, frequency of luminiferous radiation).

Instead of bodies with all the properties one perceives them to have, the “bodies” of the mathematical-perceptual world are essentially clusters of objectively measurable abstract qualities, *geometric extensions* endowed with *mass*, which measures its capacity to resist changes of state of motion, *electrical charge*, *electrical conductivity*, *thermal capacity*, *velocity*, *position*, *acceleration*, etc., all essentially numbers or

higher-order mathematical objects such as vectors.¹⁷ The mathematical-perceptual world can be further enlarged with entities that have no *direct* correspondent in perception, such as fields, forces, potentials, etc. At this point, mathematical substitutes of perceptual reality are no longer exclusively representational; they became methodological tools with which to explore and probe perceptual reality, mathematical manifolds like any others to be investigated by mathematical means like all others. Mathematical theories of the mathematical-physical world must eventually be confronted with perceptual reality, but not *directly*, with *raw* perception, but *indirectly*, with a *mathematically purified* version of perception.

The state of the world is at any point in time characterized by the values of relevant variables and how they correlate to one another. Both variables and correlations can change with the flow of time, hopefully in a lawful manner. For, infinite as it is, the world is *supposed* to be submitted to strict causal laws that can be adequately expressed only mathematically.¹⁸

The contrast with the perceptual reality of the life-world is striking. The perceptual world is a materially filled, sensorial world, a world of colors, scents, sounds, textures, and tastes. It is also a finite world, although open to a potentially infinite horizon of possibilities. Perceptual space is a sensorial, not mathematical space, and chains of causalities in the perceptual world can only be expressed in morphological (descriptive), not mathematical terms.

The constitution of a mathematical representant of the life-world is, as we have seen, a complex intentional process. It involves formal abstraction and idealization, but also selection. Of all the qualities that make the life-world, only those that can be objectively mathematized, the primary qualities, are considered worthy of inhabiting the mathematized physical world. The others, the secondary qualities, are dismissed as essentially subjective unless they can be causally related to mathematizable qualities that can, then, take their places as true objectively real qualities. A lot happens in perceptual physical reality, the only truly *real*, that do not find a way into mathematical-physics.¹⁹

Moving back from mathematical representations of the perceptual world to the perceptual world itself is not always so straightforward, even when all mathematical entities represent something in principle perceivable. Idealizations are sometimes so dramatic that mathematical representations are almost useless to handle concrete situations. Approximate techniques must then be devised – for example, linearization – and sometimes sheer brute force is preferable to sophisticate mathematical models.²⁰

5. To conclude, let's see how much of what was said can be traced back to Husserl himself. One thing he clearly saw, that the mathematical representation of perceptual reality, the first and most fundamental use of mathematics in science, on which all others depend, is a *methodological tool*, not the uncovering of the innermost aspect of physical reality.²¹ Reality, *as it appears to us*, is *perceptual reality*, which we manage to constitute out of the sensorial hyletic material.²² *Mathematical-perceptual reality is a methodological construct, not the unperceivable mathematical core of perceptual reality.*²³ Mathematical structures are not *in reality*, they only *represent*, for *strictly methodological purposes*, formal-structural aspects of reality in highly idealized form. To forget this is to cloud the applicability of mathematics in science in mystery, making it utterly incomprehensible.

Husserl was also aware of the predictive role of mathematics; in fact, it seems that this was for him the sole role mathematics plays in science: to provide more refined anticipations of experience than those allowed to perception. All this is clearly stated in *Crisis* (Husserl 1970, §9h):

“Mathematics and the mathematical science of Nature’, or still the *dressings with symbols* of symbolic-mathematical theories, contains all that that, for the expert and the cultivated men, replaces (as the objectively real and true Nature) the life-world, substituting it. It is this covering of ideas that makes us take for the true Being what is only a method – a method that is there to correct, in an infinite progression, by “scientific” anticipations, the “rough” anticipations that are originally the only that are possible in the realm of the effectively (really and

possibly) experienced in the life-world. It is this covering of ideas that renders the authentic sense of the method, formulas and theories incomprehensible, and that, in the naiveté of the method at its birth, was never understood”.

The expression “anticipation of experience” is somewhat ambiguous. Clearly, when calculating the unknown value of some magnitude in terms of others that are known by using mathematical formulas one is making a prediction, i.e. an anticipation of experience. But two things can happen here; one, all variables in the formula have known denotations. For example, computing the value of the pressure of a gas knowing its temperature and volume using the law of perfect gases. Another, some variables do not have interpretation (a semantic content), appearing in the formula only as mathematical contributions. For example, “electrical resistance” as something “in the body” accounting for the expression of the linear dependence between voltage and current intensity in *that* body. In this case, mathematics does not “predict” simply the *value* of a *known* magnitude but the *existence* of a *new* relevant magnitude, thus playing a heuristic role. Both cases can count as anticipations of experience and Husserl may have had both in mind, thus including the heuristic as part of the predictive role of mathematics, but this cannot be asserted with certainty.

Husserl also realized that mathematics has an instrumental role in science, but here he becomes a bit too conservative. He thought that with the introduction of non-representing elements in mathematical context of representation science risked meaninglessness and losing itself in symbolic alienation, away from the *possibility* of perception, the ground where knowledge must be rooted. Physical knowledge can be symbolic, he thinks, but only insofar as symbols *already have or can be given a meaning in the perceptual world*, and for strictly practical reasons. To indulge in *essentially* meaningless symbolic manipulations, involving symbols that could not, even in principle, be referred to perception, was for him to open the doors to possible falsities. Even though mathematical-physical theories involving in an essential, non-eliminable way meaningless symbols may be consistent, if these theories consistently extent theories without

meaningless symbols, Husserl claims, their observable consequences need not to be true.²⁴

However, as I believed to have shown, purely formal (non-representing) extensions of representing mathematical theories are fundamental for mathematics to cover the full spectrum of its possible scientific applications, more notably the predictive and the heuristic, and any restriction in this direction is undesirable, no matter how counterintuitive. Husserl seems to oppose such liberalism. The fundamental place he reserves for intuition, perception particularly, in his epistemology seems to block the essential (non-eliminable) use of non-intuitive, symbolic methods of knowledge.

Nonetheless, he poses no such restrictions to formal mathematics as a logical propaedeutic to knowledge. At its highest level, Husserl claims, formal logic has the task of investigating (on the formal apophantic side) formal systems in general, even invented ones, their corresponding formal domains (on the formal ontological side), and their mutual relations: theories to theories, domains to domains (a sort of universal algebra) and, as should be obvious but not explicitly mentioned by Husserl as far as I know, theories to domains (a sort of model theory). By investigating how truths migrate from theories to theories and from domains to domains through the formal-logical relations theories entertain with theories and domains with domains, we can hope, it seems, to find out conditions that *guarantee*, at least in some cases, that (in principle) perceivable consequences of theories with imaginary symbols consistently extending purely perceptual theories are actually *true*, thus providing logical-epistemological justification for the use of theories with non-eliminable “imaginary” components in science.

Husserl’s dismissal of imaginaries as possible sources of falsities in science as explicitly stated in his talks of 1901 in Göttingen and apparently implicitly admitted in *Crisis*, where it is also considered as a form of alienation, does not have, then, a strictly *formal-logical* motivation. Such a prejudice seems to emanate from an *epistemological* constraint: our knowledge of the perceptual world, the only real world, can rely on mathematical methodologies, provided, however, that we can

always “undo” abstractive and idealizing acts and move back from mathematics to perception. Mathematical representability must always be a two-way street. The primacy of intuition in Husserl’s epistemology holds firm to the very end and the call to personal responsibility still resonates in *Crisis*, his last word on the matter.

NOTES

¹ I will not address this question here but, according to Husserl, mathematical theories of perceptual reality (whether they involve imaginary entities or not, see below) are not confronted with raw perception directly, but with mathematical *substitutes* of perceptual reality. Therefore, the mathematical character of reality, as a presupposition about the essence of reality, is never put to *empirical* test. This presupposition (or hypothesis, as Husserl calls it) must, then, be either a metaphysical presupposition, as tacitly admitted in general, or, according to Husserl, a *transcendental presupposition* that goes with the intentional constitution of physical reality as an object of mathematical-physics (see Husserl 1970, §9e).

² However, as I will argue below, Husserl probably did not see such a justification as *epistemologically* valid.

³ Husserl sees in Greek geometry already a change of the original meaning of geometry.

⁴ Free fall is an important topic in *Dialogo*, where a law is stated according to which the final velocity of a body falling freely along an inclined plane depends only on its height, not inclination. From the definition of equality of velocities, Galileo arrives at the law that the times spent by two bodies falling freely along inclined planes of same height and different inclinations is in the same ratio as the lengths traveled.

⁵ Here is a much simpler example of how mathematics can reveal, once physical nature is mathematized, *quantitatively, but not qualitatively*, hitherto unknown aspects of reality: experience shows that the intensity of electric current depends linearly on the difference of electric potential (Ohm’s law), but that this relation varies with the object considered. This suggests that objects have different “electrical resistances”, the *nature* of which we may not understand, but whose *intensity* one can measure. From this notion one can define another, electrical resistivity, which depends only on the material the object is made. Theories can be devised to give these magnitudes a qualitative content (atomic theory, for example), but in these efforts mathematics can only help by imposing *formal* (in this example, quantitative) *constraints*.

⁶ Formal-abstract aspects of reality are “structural” in the sense that they have to do with *how* things relate to one another independently of *what* they are and the nature of these relations. Formal abstraction dematerializes, leaving behind only abstract form (structure).

⁷ Formal aspects of reality are *abstract* in the sense of being *ontologically dependent*: how things are related depends ontologically on there being things related in some way. Structures are abstract aspects of structured system of things, themselves somehow given or only theoretically characterized. By being *ideated* (turned into ideas) structures admit different instantiations. *Idealization* is not the same as *ideation*; idealization is a limit process; for example, a physical body and its trajectory in space taken, respectively, as a material geometrical *point* and a geometrical *line* or the quantitative relation of the mass of a body with respect to that of a standard body as given by a *single* well-determined real number.

⁸ Structural aspects of reality are, in a sense, ways of seeing, perspectives.

⁹ Perceptual reality is only another name of physical reality; physical reality is in principle always perceivable, directly or indirectly.

¹⁰ Magnitudes are countable when expressible by natural numbers and continuous when expressible by real numbers. Numbers, either natural or real, are ideal formal-abstract relations with which one can express in idealized form quantitative relations of physical reality. So-called complex and more general number concepts are not, strictly speaking, numbers, but number-like entities behaving only operationally like numbers.

¹¹ Sometimes numerical structures are used to represent aspects of physical reality without numbers expressing quantity but functioning instead as tags.

¹² Of course, explanation usually goes with predictive power, but I believe that science is fundamentally an instrument to predict and control. The history of modern science seems to show, or so I think, that putting order in the phenomenal world and being capable of making reliable predictions about it are usually preferred to having an explanation of its behavior when these tasks cannot be simultaneously fulfilled.

¹³ Mathematics itself is rooted in and born out of practices of the lifeworld (see da Silva 2017).

¹⁴ It is *because* scientific theories have infinite domains that they must, according to Husserl, be ideally *definite*, that is, finitely axiomatizable and syntactically complete.

¹⁵ Abstraction, of course, is an intentional, not a real process, either physical, which would be absurd, or mental. Abstraction is a refocusing of intentionality.

¹⁶ To ideate, on the other hand, is posit an *idea* from its instances by free variation.

¹⁷ After reducing the objects of the world to clusters of numbers it is easy to be misled into believing that they are *nothing but* numbers, a dramatic form of Pythagoreanism. This, however, is taking a product for a given, “forgetting” intentional action. Max Tegmark (Tegmark 2014) indulges lavishly in this mistake; from the purely mathematical character of mathematical-physical reality (a construct) he infers that transcendent reality (a given) is purely mathematical, taking a *method* of scientific inquiry for its *object*.

¹⁸ One may wonder at the status of such presupposition, that the world is submitted to laws. As I see it, this is neither a hypothesis nor an empirical fact but, rather, a transcendental presupposition that goes with the intentional constitution of the world as an object of scientific inquiry.

¹⁹ Therefore, as Husserl insisted, mathematical-physics cannot serve as the model for all sciences, not even for all the *natural* sciences. Phenomenology, for example, which intends to be essentially descriptive and intuitive, has not much to learn from it.

²⁰ “The point here is that when we carry out engineering in different circumstances, the way we perform mathematics changes. Often the reality is that when analytical methods become too complex, we simply resort to empirical models and simulations” (Abbot 2017, section III).

²¹ The methodological effectiveness of mathematical representations of perceptual reality can easily mislead the scientist into believing that *transcendent* reality existing out there, *prior* to being projected into the consciousness of the perceiving subject as *perceptual* reality, is *itself* a mathematical manifold. Otherwise, he thinks, the effectiveness of mathematics in science becomes a mystery. It takes a transcendental-idealist perspective to solve this “mystery” and move mathematics from metaphysics to methodology, where it belongs.

²² The constitution of perceptual reality, as we have seen, is not properly intentional since it is not a fully conscious process. Perceptual reality is constituted as transcendent reality “filters through” the senses and the sensorial impressions are “interpreted” by proto-intentional psychophysical systems that give them perceptual sense.

²³ Husserl is meridionally clear about this, the mathematical world is not a given, but a rational construct (see Husserl 1970, §9b)

²⁴ Although Husserl is not so explicit in *Crisis*, he leaves no margin for doubt as to the role of the “imaginary” in science, or at least in mathematics, in a couple of talks he delivered in Göttingen in 1901 (Husserl 1970, pp. 430-51): maybe to facilitate calculations that must, however, at least in principle, be possible without them. Non-denoting terms cannot have an *essential* role in science (see da Silva 2010). He never disowned such a view, which appears in later texts as well, and there is no reason why he would not have extended it to the whole of science, not only contentual, interpreted mathematics (*formal* mathematics, on the other hand, as a chapter of formal ontology is another matter; it is, in a sense, “imaginary” mathematics and need not care about interpretations).

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